Revista de Matemática: Teoría y Aplicaciones 2011 **18**(1) : 111–120
cimpa – ucr   issn: 1409-2433

A NON-STANDARD GENERATING FUNCTION FOR CONTINUOUS DUAL q-HAHN POLYNOMIALS

UNA FUNCIÓN GENERATRIZ NO ESTÁNDAR PARA POLINOMIOS q-HAHN DUALES CONTINUOS

Mesuma Atakishiyeva*   Natig Atakishiyev†

Received: 18 Feb 2010; Revised: 22 Oct 2010; Accepted: 23 Nov 2010

Abstract

We study a non-standard form of generating function for the three-parameter continuous dual q-Hahn polynomials $p_n(x; a, b, c \,|\, q)$, which has surfaced in a recent work of the present authors on the construction of lifting q-difference operators in the Askey scheme of basic hypergeometric polynomials. We show that the resulting generating function identity for the continuous dual q-Hahn polynomials $p_n(x; a, b, c \,|\, q)$ can be explicitly stated in terms of Jackson's q-exponential functions $e_q(z)$.

*Facultad de Ciencias, Universidad Autónoma del Estado de Morelos, C.P. 62250 Cuernavaca, Morelos, México. E-Mail: mesuma@servm.fc.uaem.mx

†Instituto de Matemáticas, Unidad Cuernavaca, Universidad Nacional Autónoma de México, C.P. 62251 Cuernavaca, Morelos, México. E-Mail: natig@matcuer.unam.mx

Keywords: q-scheme of Askey, generating function, q-exponential function of Jackson, dual q-Hahn polynomials.

Resumen

Estudiamos una forma no estándar de la función generatriz para una familia de polinomios duales continuos q-Hahn de tres parámetros $p_n(\,x;\,a,b,c\,|\,q\,)$, que han surgido en un trabajo reciente de los autores en la construcción de operadores elevadores en q-diferencias del esquema de Askey de polinomios básicos hipergeométricos. Demostramos que la función generatriz identidad resultante para los polinomios q-Hahn duales continuos $p_n(x;a,b,c\,|\,q)$ puede ser expresada explícitamente en términos de las funciones q-exponenciales de Jackson $e_q(z)$.

Palabras clave: esquema q de Askey, función generatriz, polinomios duales q-Hahn, función q-exponencial de Jackson.

Mathematics Subject Classification: 33D45, 39A70, 47B39.

1 Introduction

We start out by quoting a metaphorical passage by H.S.Wilf about the main subject of the present work: "A generating function is a clothesline on which we hang up a sequence of numbers for display" [1]. Indeed, a power series $f(t) = \sum_{n=0}^{\infty} a_n\, t^n$, whose coefficients give the sequence $\{a_0, a_1, \ldots\}$, is sometimes said to "enumerate" a_n (see [2], p.85). There are many beautiful generating functions in number theory: two particularly nice examples are given by $f(t) = 1/\prod_{k=1}^{\infty}(1-t^k) = \sum_{n=0}^{\infty} p\,(n)\, t^n = 1 + t + 2t^2 + 3t^3 + \ldots$ for the partition function $p\,(n)$, and $f(t) = t/(1-t-t^2) = \sum_{n=0}^{\infty} F_n\, t^n = t + t^2 + 2t^3 + 3t^4 + \ldots$ for the Fibonacci numbers F_n.

The next natural step is to extend this notion of generating function by allowing the coefficients a_n to be dependent on another variable, x. Most of the special functions of mathematical physics may be defined by a generating function of this more general form

$$g(t,x) := \sum_{n=0}^{\infty} c_n\, t^n p_n(x)\,, \tag{1.1}$$

where $p_n(x)$ are some polynomials in the variable x and the c_n are constants. So in the spirit of the quotation from [1] we used earlier, in generating function (1.1) "a sequence of polynomials $p_n(x)$, $n = 0, 1, 2, \ldots$, is hung up for display", rather than a sequence of numbers. For instance, the Hermite polynomials $H_n(x)$ have the generating function of the form

$$\exp\Big(2xt - t^2\Big) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, H_n(x)\,. \tag{1.2}$$

It is worth noting that the real beauty of the generating function (1.2) is that it actually *compactly embodies* a large amount of information about the Hermite polynomials $H_n(x)$: one can readily derive from (1.2) the Rodrigues formula $H_n(x) = (-1)^n\, e^{x^2}\big(\frac{d}{dx}\big)^n e^{-x^2}$, the three-term recurrence relation $2x\,H_n(x) = H_{n+1}(x) + 2n\,H_{n-1}(x)$, the differentiation formula $H_n^{'}(x) = 2n\,H_{n-1}(x)$, the second-order differential equation $H_n^{''}(x) - 2x\,H_n^{'}(x) + 2n\,H_n(x) = 0$ for the Hermite polynomials $H_n(x)$, and so on.

We also recall that basic hypergeometric polynomials from the Askey q-scheme [3] are associated with so-called q-extensions of generating function (1.1), which depend on the "deformation" parameter q. For instance, the continuous q-Hermite polynomials $H_n(x|\,q)$ of Rogers have a generating function of the form

$$\sum_{n=0}^{\infty} \frac{H_n(x|\,q)}{(q\,;q)_n}\, t^n = e_q\Big(t\,e^{\mathrm{i}\theta}\Big)\, e_q\Big(t\,e^{-\mathrm{i}\theta}\Big), \qquad x = \cos\theta, \qquad |\,t| < 1, \tag{1.3}$$

where Jackson's q-exponential function $e_q(z)$ is defined as

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = (z;q)_\infty^{-1}\,, \qquad |z| < 1\,. \tag{1.4}$$

This short work is aimed at the study of a non-standard form of generating function for the three-parameter continuous dual q-Hahn polynomials $p_n(x;a,b,c\,|\,q)$, which has surfaced in our recent work [4] on the construction of lifting q-difference operators in the Askey scheme of basic hypergeometric polynomials. In [4] we have determined first an explicit form of a q-difference operator that lifts the continuous q-Hermite polynomials $H_n(x|\,q)$ of Rogers up to the continuous big q-Hermite polynomials $H_n(x;a|\,q)$ on the next level in the Askey scheme of basic hypergeometric polynomials. This operator is defined as Exton's q-exponential function (introduced in [5] and studied in detail in [6]–[9]) $\varepsilon_q(a_q D_q)$,

$$\varepsilon_q(z) := \sum_{k=0}^{\infty} q^{\,k(k+1)/4}\, \frac{z^k}{(q;q)_k} = {}_1\phi_1\Big(0\,; -q^{1/2}; q^{1/2}, -q^{1/2} z\Big)\,,$$

in terms of the Askey–Wilson divided q-difference operator D_q and it represents a particular q-extension of the standard shift operator $\exp\Big(a\frac{d}{dx}\Big)$. We have demonstrated next that one may move two steps more upwards in order first to reach the Al-Salam–Chihara family of polynomials $Q_n(x;a,b\,|\,q)$ and then the continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$. In both of these cases lifting operators turn out to be convolution-type products of two and three, respectively, one-parameter q-difference operators of the same type $\varepsilon_q(a_q D_q)$ as at the initial step.

To achieve our goal in this work, we have collected in section 2 some background facts about a family of the continuous dual q-Hahn polynomials $p_n(x;a,b,c\,|\,q)$, which are then used in section 3 in order to find an explicit form of a non-standard generating function identity for the continuous dual q-Hahn polynomials in terms of Jackson's q-exponential functions $e_q(z)$. The key to our approach lies in the essential use of the Askey–Wilson formalism of connection coefficients [10], providing an explicit link between the continuous dual q-Hahn polynomials $p_n(x;a,b,c\,|\,q)$ and the Al-Salam–Chihara polynomials $Q_n(x;a,b|\,q)$.

Throughout this exposition we employ standard notations of the theory of special functions (see, for example, [11]–[13]).

2 Continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$

The continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$ from the third level in the Askey q-scheme depend on three parameters a, b and c (in addition to the base q) and they are explicitly given by means of the formula

$$p_n(x;a,b,c|\,q) := \frac{(ab,ac\,;q)_n}{a^n}\,{}_3\phi_2\left(\begin{array}{c} q^{-n},\, a\,e^{\mathrm{i}\theta},\, a\,e^{-\mathrm{i}\theta} \\ ab,\, ac \end{array}\middle|\, q\,;q\right)$$

$$\equiv \frac{(ab,ac\,;q)_n}{a^n}\sum_{k=0}^{n}\frac{(q^{-n},a\,e^{\mathrm{i}\theta},a\,e^{-\mathrm{i}\theta};q)_n}{(ab,ac,q;q)_n}\,q^{\,n}\,, \qquad x=\cos\theta\,, \tag{2.1}$$

where $(z;q)_n$ is the q-shifted factorial, $(z;q)_0=1$, $(z;q)_n=\prod_{k=0}^{n-1}(1-zq^k)$ for $n=1,2,3,\ldots$, and we have used the conventional short notation $(a_1,a_2,\ldots,a_k;q)_n := \prod_{j=1}^{k}(a_j;q)_n$ for products of q-shifted factorials. If a, b and c are real, or one of them is real and the other two are complex conjugates and $\max\left(|\,a|,|\,b|,|\,c|\,\right)<1$), the polynomials $p_n(x;a,b,c|\,q)$ are orthogonal on the finite interval $-1\le x:=\cos\theta\le 1$,

$$\frac{1}{2\pi}\int_{-1}^{1} p_m(x;a,b,c|\,q)\,p_n(x;a,b,c|\,q)\,w(x;a,b,c|\,q)\,dx \;=\; h_n\,\delta_{mn}\,,$$

$$h_n = \left(q^{n+1},\, abq^n,\, acq^n,\, bcq^n\,;q\right)_\infty^{-1}, \tag{2.2}$$

with respect to the weight function (*cf* formula (3.3.2) on p.69 in Ref.[3])

$$w(x;a,b,c|\,q) := \left| e_q\Big(a\,e^{\mathrm{i}\theta}\Big)\, e_q\Big(b\,e^{\mathrm{i}\theta}\Big)\, e_q\Big(c\,e^{\mathrm{i}\theta}\Big)\right|^2 w(x|\,q)\,, \tag{2.3}$$

where $w(x|\,q)$ is the weight function for the continuous q-Hermite polynomials of Rogers $H_n(x|\,q)$,

$$w(x|\,q) := \frac{1}{\sin\theta}\left(e^{2\mathrm{i}\theta};q\right)_\infty \left(e^{-2\mathrm{i}\theta};q\right)_\infty. \tag{2.4}$$

To avoid any confusion of notations, we emphasize that the weight functions $w(x;a,b,c|\,q)$ and $w(x|\,q)$, defined by (2.3) and (2.4), respectively, are the same as $\widetilde{w}(x;a,b,c|\,q)$ and $\widetilde{w}(x|\,q)$ in, for example, Ref. [3]. It should be noted that in what follows we restrict ourselves to the case when the parameters a, b and c are real and $\max(|\,a|,|\,b|,|\,c|\,) < 1$.

The continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$ are symmetric with respect to the parameters a, b and c; when one of them is equated to zero, they reduce to the Al-Salam–Chihara polynomials $Q_n(x;a,b|\,q)$, that is, $p_n(x;a,b,0|\,q) = Q_n(x;a,b|\,q)$. For general values of the parameters a, b and c these two families of q-polynomials are interrelated as [10]

$$p_n(x;a,b,c|\,q) = \sum_{k=0}^{n} C_{n,\,k}\, Q_k(x;b,c|\,q)\,, \tag{2.5}$$

where the connection coefficients

$$C_{n,\,k} = q^{\,k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q c^{\,k-n}\,\frac{(ac,bc\,;q)_n}{(ac,bc\,;q)_k}\,{}_2\phi_1\left(\left.\begin{matrix} q^{\,k-n},\,0 \\ ac\,q^{\,k}\end{matrix}\,\right|\, q\,;q\right). \tag{2.6}$$

The basic ${}_2\phi_1$-polynomial on the right-hand side of (2.6) can be evaluated as a special case of the Chu–Vandermonde q-sum for ${}_2\phi_1(q^{-n},b\,;c\,;q,q)$ with a vanishing parameter b ([11], formula (1.5.3) on p.14, see also [14]), that is,

$${}_2\phi_1(q^{-n},0\,;c\,;q,q) \;=\; q^{n(n-1)/2}\,\frac{(-c)^n}{(c;q)_n}\,.$$

Thus (2.5) reduces to the relation

$$p_n(x;a,b,c|\,q) = \sum_{k=0}^{n} q^{\frac{(n-k)(n-k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-\,a)^{n-\,k}\,\frac{(b\,c\,;q)_n}{(b\,c\,;q)_k}\,Q_k(x;b,c|\,q)$$

$$\equiv \sum_{m=0}^{n} q^{\,m(m-\,1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q (-\,a)^m\,\frac{(b\,c\,;q)_n}{(b\,c\,;q)_{n-\,m}}\,Q_{n-\,m}(x;b,c|\,q)$$

$$= \sum_{m=0}^{n} \left(a\,b\,c\,q^{\,n-1}\right)^m \left(\frac{q^{1-n}}{b\,c}\,;q\right)_m \begin{bmatrix} n \\ m \end{bmatrix}_q Q_{n-m}(x;b,c|\,q)$$

$$= \sum_{m=0}^{n} q^{m(m-1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q (-a)^m \left(b\,c\,q^{\,n-1}\,;q^{-1}\right)_m Q_{n-m}(x;b,c|\,q)\,, \quad (2.7)$$

where at the last step we employ the inversion formula

$$\left(z;q^{-1}\right)_n = q^{-\,n(n-1)/2}\,(-z)^n \left(z^{-1};q\right)_n$$

with respect to the transformation of the base $q \Rightarrow 1/q$. This key connection coefficient link between the continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$ and the Al-Salam–Chihara polynomials $Q_n(x;a,b|\,q)$ will be essentially used in the next section.

In the case when any two of the parameters a, b and c vanish, the continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$ reduce to the continuous big q-Hermite polynomials $H_n(x;a|\,q)$, *i.e.*, $p_n(x;a,0,0|\,q) = H_n(x;a|\,q)$, finally, when all three parameters vanish they coincide with the continuous q-Hermite polynomials $H_n(x|\,q)$, that is, $p_n(x;0,0,0|\,q) = H_n(x|\,q)$. We note in passing that the generating function (2.5) reduces to the (1.3) in the particular case when $a = b = c = 0$.

There are four standard generating function identities for the continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$ in [3] (see formulas (3.3.13)–(3.3.16)); one example to illustrate their nature is of the form

$$\frac{(c\,t\,;q)_\infty}{(t\,e^{\mathrm{i}\theta};q)_\infty}\,{}_2\phi_1\left(\begin{array}{c} a\,e^{\mathrm{i}\theta},\,b\,e^{\mathrm{i}\theta} \\ ab \end{array}\middle|\,q\,;t\,e^{-\mathrm{i}\theta}\right) = \sum_{n=0}^{\infty}\frac{p_n(x;a,b,c|\,q)}{(ab,q\,;q)_n}\,t^{\,n}\,, \quad (2.8)$$

where $x = \cos\theta$ and $|\,t| < 1$ (see (3.3.13) on p.70 in [3]). Observe that when one of the parameters a, b and c vanish, the generating function (2.8) reduces to the well-known generating function

$$\sum_{k=0}^{\infty}\frac{t^{\,k}}{(q;q)_k}\,Q_k(x;b,c|\,q) = (bt,ct\,;q)_\infty\,e_q\left(te^{\mathrm{i}\theta}\right)e_q\left(te^{-\mathrm{i}\theta}\right),\ |t|<1, \quad (2.9)$$

for the Al-Salam–Chihara polynomials $Q_n(x;a,b\,|\,q)$ ((3.8.13),p.81, [3]).

3 Non-standard generating function for $p_n(x;a,b,c|\,q)$

We now turn to the determination of a non-standard generating function for the continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$ of the form

$$f(t,x\,;a,b,c|\,q) := \sum_{n=0}^{\infty}\frac{p_n(x;a,b,c|\,q)}{(abc\,t,q\,;q)_n}\,t^{\,n}\,, \quad (3.1)$$

which differs from standard ones by a t-dependent denominator factor $(abc\,t;q)_n$ in the right side of (3.1) (*cf.* formulas (3.3.13)–(3.3.16)) in [3]). So we begin this section with an evaluation of this curious, and perhaps surprising, generating function (3.1), which is of considerable interest in itself. Evidently, since the $p_n(x;0,0,0|\,q)$ coincide with the $H_n(x|\,q)$, the generating function (3.1) for $a=b=c=0$ should reduce to the right side of the identity (1.3).

The first step in evaluating (3.1) is to substitute the connection relation (2.7) for the continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$ in terms of the Al-Salam–Chihara polynomials $Q_n(x;a,b|\,q)$ into the right side of (3.1). This gives the relation

$$\begin{aligned} f(t,x\,;a,b,c|\,q) &= \sum_{n=0}^{\infty}\frac{t^{\,n}}{(abc\,t,q\,;q)_n}\sum_{m=0}^{n} q^{m(m-1)/2}\begin{bmatrix} n \\ m \end{bmatrix}_q \\ &\quad\times(-a)^m\left(b\,c\,q^{\,n-1}\,;q^{-1}\right)_m Q_{n-m}(x;b,c|\,q) \\ &= \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}\,(bc\,;q)_n}{(abct,q\,;q)_n}\,t^n\sum_{k=0}^{n} q^{\,k(k+1-2n)/2}\begin{bmatrix} n \\ k \end{bmatrix}_q \\ &\quad\times\frac{(-\,a)^{n-\,k}}{(bc\,;q)_k}\,Q_k(x;b,c|\,q)\,, \end{aligned} \tag{3.2}$$

where we have inverted the order of summation with respect to the index m (*i.e.*, $k=n-m$) and used the identities (see, for example, (I.8) and (I.10) on page 233 in [11])

$$(z;q)_{n-\,k} \;=\; \frac{(z;q)_n}{(q^{1-\,n}/z;q)_k}\left(-\,\frac{q}{z}\right)^k q^{\,k(k-1-2n)/2}\,,$$

$$(zq^{-\,n};q)_n \;=\; (q/z;q)_n\left(-\,\frac{z}{q}\right)^n q^{\,-\,n(n-1)/2}\,.$$

The next step is to interchange the order of summation with respect to the indices n and k, which yields

$$\begin{aligned} f(t,x\,;a,b,c|\,q) &= \sum_{k=0}^{\infty} q^{\,k(k+1)/2}\,\frac{Q_n(x;b,c|\,q)}{(-\,a)^k\,(bc,q\,;q)_k} \\ &\quad\times\sum_{n=k}^{\infty} q^{n(n-1-2k)/2}\,\frac{(bc\,;q)_n\,(-\,at)^n}{(abct\,;q)_n\,(q;q)_{n-k}} \\ &= \sum_{k=0}^{\infty}\frac{Q_n(x;b,c|\,q)}{(-\,a)^k\,(bc,q\,;q)_k}\sum_{m=0}^{\infty} q^{m(m-1)/2}\,\frac{(bc\,;q)_{m+k}\,(-\,at)^{m+k}}{(abct\,;q)_{m+k}\,(q;q)_m} \end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{Q_n(x;b,c|\,q)}{(abct,q\,;q)_k}\, t^{\,k} \sum_{m=0}^{\infty} q^{m(m-1)/2}\, \frac{(bcq^{\,k}\,;q)_m\,(-\,at)^m}{(abctq^{\,k},q\,;q)_m}\,, \tag{3.3}$$

where at the last step we employ the identity $(A\,;q)_{m+k}=(A\,;q)_k\,(A\,q^k\,;q)_m$ twice. The sum over the index m in (3.3) represents a basic hypergeometric series

$${}_1\phi_1\Big(bcq^{\,k};\; abctq^{\,k}; q, at\Big)$$

and can be now evaluated by the ${}_1\phi_1$-sum (see 1.6(ii) on p.26 in [11])

$${}_1\phi_1\Big(A;\; B; q, B/A\Big) \;=\; \frac{(B/A\,;q)_\infty}{(B\,;q)_\infty}$$

with $A=bcq^{\,k}$ and $B=abctq^{\,k}$. Consequently, one arrives at the required relation

$$\begin{aligned} f(t,x\,;a,b,c|\,q) &= \sum_{k=0}^{\infty} \frac{Q_k(x;b,c|\,q)}{(abct,q\,;q)_k}\, t^{\,k}\, \frac{(at\,;q)_\infty}{(abctq^{\,k}\,;q)_\infty} \\ &= \frac{(at\,;q)_\infty}{(abct\,;q)_\infty} \sum_{k=0}^{\infty} \frac{t^{\,k}}{(q\,;q)_k}\, Q_k(x;b,c|\,q) \\ &= \frac{(at,bt,ct\,;q)_\infty}{(abct\,;q)_\infty}\, e_q\left(te^{\mathrm{i}\theta}\right) e_q\left(te^{-\mathrm{i}\theta}\right)\,, \end{aligned} \tag{3.4}$$

upon making use of the known generating function identity (2.9) for the Al-Salam–Chihara polynomials $Q_n(x;a,b|\,q)$. Thus, from (3.4) one deduces that a non-standard generating function identity for the continuous dual q-Hahn polynomials $p_n(x;a,b,c|\,q)$ is of the form

$$\sum_{n=0}^{\infty} \frac{p_{\,n}(x;a,b,c|\,q)}{(abct,q\,;q)_n}\, t^{\,n} = \frac{(at,bt,ct\,;q)_\infty}{(abct\,;q)_\infty}\; e_q\Big(t\,e^{\mathrm{i}\theta}\Big)\, e_q\Big(t\,e^{-\mathrm{i}\theta}\Big)\,,$$

$$x=\cos\theta\,, \qquad |\,t|<1\,. \tag{3.5}$$

4 Concluding remarks

In our recent article [4] we were driven by the quest to construct explicitly q-difference operators that lift successively the continuous q-Hermite polynomials $H_n(x|\,q)$ of Rogers up first to the continuous big q-Hermite polynomials $H_n(x;a|\,q)$, then to the Al-Salam–Chihara polynomials $Q_{\,n}(x;a,b|\,q)$, and, finally, to the continuous dual q-Hahn polynomials $p_{\,n}(x;a,b,c|\,q)$ in the Askey scheme of basic hypergeometric polynomials. In the course of attempting to take the same line of reasoning a step further and reach the

Askey–Wilson polynomials $p_n(x; a, b, c, d|\, q)$ on the top level in the Askey q-scheme, it has become clear that one faces the problem of evaluating the non-standard generating function $f(t, x\,; a, b, c|\, q)$. So a resolution of this problem is (3.5) and we are now in a position to formulate a follow-up to [4]. Work on presenting this development is in progress.

Obviously, (3.5) is of considerable mathematical interest in itself and it becomes of central importance to find out whether there are similar types of non-standard generating functions for other families of basic hypergeometric polynomials from the Askey q-scheme.

Acknowledgements

The participation of MKA in this work has been supported by the Proyecto de la Red de Cuerpos Académicos "Análisis, Teoría de Operadores Moderna y sus Aplicaciones en la Física Matemática" (2009-2010), whereas NMA has been supported by the DGAPA-UNAM IN105008-3 and SEP-CONACYT 79899 projects "Óptica Matemática".

References

[1] Wilf, H. (2004) *Generatingfunctionology*. Cambridge University Press, Cambridge.

[2] Hardy, G. (1999) *Ramanujan: Twelwe Lectures on Subjects Suggested by His Life and Work*. Chelsea, New York.

[3] Koekoek, R.; Swarttouw, R. (1998) "The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue", Report 98–17, Delft University of Technology, Delft.

[4] Atakishiyeva, M.; Atakishiyev, N. (2010) "On lifting q-difference operators in the Askey scheme of basic hypergeometric polynomials", *J. Phys. A: Math. Theor.* **43**(14): 145201–145218.

[5] Exton H. (1983) *q-Hypergeomteric Functions and Applications*. Ellis Horwood, Chichester.

[6] Atakishiyev, N.; Suslov, S. (1992) "Difference Hypergeometric Functions", in: A. Gonchar & E. Saff (Eds.) *Progress in Approximation Theory: An International Perspective*, Springer-Verlag, New York, Berlin: 1–35.

[7] Nelson, C.; Gartley, M. (1994) "On the zeros of the q-analogue exponential function", *J. Phys. A: Math. Gen.* **27**(11): 3857–3881.

[8] Rahman, M. (1995) "The q-exponential functions, old and new", in: A. Sissakian & G. Pogosyan (Eds.) *Proceedings of the International Workshop on Finite Dimensional Integrable Systems*, Joint Institute for Nuclear Research, Dubna, Russia: 161–170.

[9] Atakishiyev, N. (1996) "On a one-parameter family of q-exponential functions", *J. Phys. A: Math. Gen.* **29**(10): L223–L227.

[10] Askey, R.; Wilson, J. (1985) "Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials", *Mem. Am. Math. Soc.* **54**(319): 1–55.

[11] Gasper, G.; Rahman, M. (2004) *Basic Hypergeometric Functions.* Cambridge University Press, Cambridge.

[12] Andrews, G.; Askey, R.; Roy, R. (1999) *Special Functions.* Cambridge University Press, Cambridge.

[13] Ismail, M. (2005) *Classical and Quantum Orthogonal Polynomials in One Variable.* Cambridge University Press, Cambridge.

[14] Atakishiyev, N. (1997) "Fourier–Gauss transforms of the Askey–Wilson polynomials", *J. Phys. A: Math. Gen.* **30**(24): L815–L820.