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EXISTENCE CONDITIONS FOR k-BARYCENTRIC OLSON CONSTANT

CONDICIONES DE EXISTENCIA PARA LA CONSTANTE DE OLSON k-BARICÉNTRICA

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Abstract

Let $(G, +)$ be a finite abelian group and $3 \leq k \leq |G|$ a positive integer. The k-barycentric Olson constant denoted by $BO(k, G)$ is defined as the smallest integer ℓ such that each set A of G with $|A| = \ell$ contains a subset with k elements $\{a_1, \ldots, a_k\}$ satisfying $a_1 + \cdots + a_k = ka_j$ for some $1 \leq j \leq k$. We establish some general conditions on G assuring the existence of $BO(k, G)$ for each $3 \leq k \leq |G|$. In particular, from our results we can derive the existence conditions for cyclic groups and for elementary p-groups $p \geq 3$. We give a special treatment over the existence condition for the elementary 2-groups.

Keywords: finite abelian group; zero-sum problem; baricentric-sum problem; Davenport constant; k-barycentric Olson constant.

Resumen

Sean $(G, +)$ un grupo abeliano finito y $3 \leq k \leq |G|$ un entero positivo. La constante de Olson k-baricéntrica, denotada por $BO(k, G)$, se define como el menor entero positivo ℓ tal que todo conjunto A de G con $|A| = \ell$ contiene un subconjunto con k elementos $\{a_1, \ldots, a_k\}$ que satisface $a_1 + \cdots + a_k = ka_j$ para algún $1 \leq j \leq k$. Establecemos algunas condiciones generales sobre G asegurando la existencia de $BO(k, G)$ para cada $3 \leq k \leq |G|$. En particular, a partir de nuestros resultados podemos determinar las condiciones de existencia para los grupos cíclicos y para los p-grupos elementales con $p \geq 3$. Damos un tratamiento especial a la condición de existencia para los 2-grupos elementales.

Palabras clave: grupos abelianos finitos; problemas de suma-cero; problemas de suma baricéntricas; constante de Davenport; constante k-baricéntrica de Olson.

Mathematics Subject Classification: 11B30, 11B75, 94B05, 94B65, 51E22.

1 Introduction

We recall some standard terminology and notation. We denote by $\mathbb{N}$ the positive integers and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For abelian groups, we use additive notation and we denote the neutral element by 0. For $n \in \mathbb{N}$, let C_n denotes a cyclic group of order n. For each finite abelian group there exists $1 < n_1 \mid \cdots \mid n_r$ such that $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$. The integer n_r is called the exponent of G, denoted $\exp(G)$. The integer r is called the rank of G, denoted $r(G)$. For a prime p, the p-rank of G, denoted $r_p(G)$, is the smallest number i such that n_i is divisible by p. For a prime number p we denote by $\mathbb{F}_p$ the field with p elements.

We say that G is a p-group if its exponent is a prime power and we say that G is an elementary p-group if the exponent is a prime (except for the trivial group). Let G be an abelian finite group. The sumset of two subsets A and B of G will be denoted by $A + B = \{a + b : a \in A \ \wedge \ b \in B\}$. We denote the sum of the elements of a subset S of G by $\sigma(S)$. Furthermore, for an integer k, let $\sum_k(A) = \{\sigma(B) : B \subseteq A \ \wedge \ |B| = k\}$. Finally, for t an integer, we denote by $t \cdot A$ the set of multiples $t \cdot A = \{ta : a \in A\}$.

For a finite abelian group $(G, +)$ and $3 \leq k \leq |G|$ a positive integer, the k-barycentric Olson constant denoted by $BO(k, G)$ is the smallest ℓ such that each set A with $|A| = \ell$ over G has a subset with k elements $\{a_1, \dots, a_k\}$ satisfying $a_1 + \cdots + a_k = ka_j$ for some $1 \leq j \leq k$. This set with k elements is called a k-barycentric set and a_j is called its barycenter. Notice that a k-barycentric set can be written as a weighted zero-sum set that is:

$$a_1 + \cdots + (1 - k)a_j + \cdots + a_k = 0.$$

So that the k-barycentric Olson constant can be seen as a classical example of a weighted zero-sum constant over a finite abelian group. This constant together with related invariants have been studied in the literature [5, 6]. The aim of the present work is to establish conditions on G for the existence of $BO(k, G) \leq |G|$ for each $3 \leq k \leq |G|$. That is to say, for each $3 \leq k \leq |G|$ there exists a k-barycentric set.

Existence conditions of the k-barycentric Olson constant with $3 \leq k \leq |G|$ were initially considered in [14] with the study on cyclic groups using the Orbits Theory. In [13] Ordaz, Plagne and Schmid researched on the existence conditions of $BO(k, G)$ with $|G| - 2 \leq k \leq |G|$ over finite abelian groups G in general; their results were Lemma 1 and Proposition 1. In case there are no k-barycentric sets in G we write $BO(k, G) = |G| + 1$.

Lemma 1 *([13] , Lemma 3.1) Let G be a finite abelian group. Then*

$$\sigma(G) = \begin{cases} b^* & \text{if } r_2(G) = 1 \text{ and } b^* \text{ denote the only element with order } 2, \\ 0 & \text{in other case.} \end{cases}$$

Hence we have that:

$$BO(|G|, G) = \begin{cases} |G| + 1 & \text{if } r_2(G) = 1, \\ |G| & \text{in other case.} \end{cases}$$

The following result gives the values of $BO(|G| - 1, G)$ and $BO(|G| - 2, G)$.

Proposition 1 *([13] , Proposition 3.2) Let G be a finite abelian group. Then for $|G| \geq 2$, we have:*

$$BO(|G|-1, G) = \begin{cases} |G|-1 & \textit{if } r_2(G) = 1, \\ |G|+1 & \textit{in other case.} \end{cases}$$

and for $|G| \geq 3$, we have:

$$BO(|G|-2, G) = \begin{cases} |G|-2 & \textit{if } |G| \textit{ is odd,} \\ |G|+1 & \textit{if } \exp(G) = 2 \textit{ or } |G| = 4, \\ |G|-1 & \textit{in other case.} \end{cases}$$

In the Lemma 1 is determine the conditions of existence of $BO(|G|, G)$ and in the Proposition 1 is determine the conditions of existence of $BO(k, G)$ with $|G| - 2 \leq k \leq |G| - 1$.

In the same order of ideas of the above results, the main goal of our paper is to show that the finite abelian groups G with $r_2(G) = 0$ and the finite abelian groups G with $r_2(G) = 1$ contain a k-barycentric set for each $3 \leq k \leq |G| - 3$. Notice that the cyclic groups C_n are members of these groups since $r_2(C_n) = 0$ if and only if n is odd and $r_2(C_n) = 1$ if and only if n is even. Similarly, elementary p-groups with $p \neq 2$, are members of the above groups since $r_2(C_p^m) = 0$. In consequence our results solve completely the existence conditions of the k-barycetric Olson constant, for cyclic groups and for elementary p-groups. It is clear that the elementary 2-groups are outside the above groups and then we have a special consideration for its existence conditions for $BO(k, C_2^m)$. As a second goal in our investigation, for some G and k, we give an exact value for $BO(k, G)$ when it exists. For example, we show that $BO(|G| - 3, G) = |G| - 2$ for the abelian groups G with $r_2(G) = 1$, $|G| \geq 8$ and non multiple of 3. Moreover, we show that $BO(3^m - 3, C_3^m) = 3^m - 2$, in this case $r_2(C_3^m) = 0$.

The organization of the paper besides this introduction and the conclusion, is as follows: a first section on preliminaries, a second section on existence conditions for general finite abelian groups and finally, a third section on some existence conditions for elementary 2-groups.

2 Preliminaries

In this section we give some previous and useful results.

Remark 1 *Let G be a finite abelian group. Then*

i. $r_2(G) = 0$ *if and only if* $|G|$ *is odd.*

ii. $r_2(G) = 1$ *implies that* $|G|$ *is even. Let* $b^* \in G$ *be the only element of order* 2. *It is clear that for cyclic groups we have the equivalence* $r_2(C_n) = 1$ *if and only if* n *is even. Also we have that* $r_2(C_p^m) = 0$ *for* $p \neq 2$. *Moreover, if* $t = r_2(G) \geq 1$, *then* $|G|$ *is even and* G *has* $2^t - 1$ *elements of order* 2.

Proposition 2 *Let G be a finite abelian group with $|G| \geq 8$ such that $r_2(G) = 1$ and $3 \nmid |G|$. Then.*

i. $-3 \cdot G = G$.

ii. Let $a \in G$ *and* $S_a = \{x \in G : 2x = a\}$. *Then* $|S_a| \leq 2$.

Proof. i. Let $\phi : G \to -3 \cdot G$ be given by $\phi(a) = -3a$ where $-3 \cdot G = \{3(-a) : a \in G\}$. Let $y = 3(-a) \in G$, then exits $a \in G$ such that $\phi(a) = -3a = 3(-a) = y$, therefore ϕ is surjective. Assuming that $\phi(a_1) = \phi(a_2)$, then $-3a_1 = -3a_2$, so that, $3(a_1 - a_2) = 0$. Since $3 \nmid |G|$, then $a_1 = a_2$, i.e., ϕ is injective. Then $|G| = |-3 \cdot G|$. Since $-3 \cdot G \subseteq G$ and G is finite, then $-3 \cdot G = G$.

ii. Assuming we have three different elements $a_1, a_2, a_3 \in S_a$, then $2a_1 = 2a_2$ and $2a_1 = 2a_3$, in consequence $2(a_1 - a_2) = 0$ and $2(a_1 - a_3) = 0$.

Since a_1, a_2, a_3 are different, then $a_1 - a_2 = b^*$ and $a_1 - a_3 = b^*$, where b^* is the only element of order 2 in G. Hence $a_2 = a_3$, contradiction. So that $|S_g| \leq 2$. ■

We have the following result:

Proposition 3 *If $m \geq 2$, then $3^m - 2 \leq BO(3^m - 3, C_3^m)$.*

Proof. Let $A = C_3^m \setminus \{-a, -b, 0\}$ be a $(3^m - 3)$-subset over C_3^m with $a + b \neq 0$. Since $\sigma(C_3^m) = 0$ and $\sigma(C_3^m) = \sigma(A) + \sigma(A^c)$ where $A^c = \{-a, -b, 0\}$, then $\sigma(A) = -\sigma(A^c) \Rightarrow \sigma(A) = -(-a - b + 0) \Rightarrow \sigma(A) = a + b \neq 0$. Moreover we have that $(3^m - 3)a = (3^m - 3)(3a) = (3^{m-1} - 1)0 = 0$ for all $a \in A \subset C_3^m$ since $3x = 0$ for all $x \in C_3^m$. Therefore, there exists a $(3^m - 3)$-subset A over C_3^m such that $\sigma(A) \neq (3^m - 3)a$ for all $a \in A$, i.e., $3^m - 2 \leq BO(3^m - 3, C_3^m)$. ■

We need the following result:

Proposition 4 *Let A be a k-subset of C_2^m such that $3 \leq k \leq 2^m$.*

i. *If k is even, then A is a k-barycentric set if and only if $\sigma(A) = 0$.*

ii. *If k is odd, then A is a k-barycentric set if and only if $\sigma(A) \in A$.*

iii. *Let $A^c = C_2^m \setminus A$ the complement of A, then $|A| = 2^m - |A^c|$.*

iv. $\sigma(A) = \sigma(A^c)$.

v. $0 \notin \sum_2 C_2^m$.

Proof. It follows directly. ■

The following lemma guarantees the existence of k-sets of zero-sum with $4 \leq k \leq \frac{|G|}{2} - 1$ in a finite abelian group.

Lemma 2 *([3] , Lemma 7.1) Let G be a finite abelian group de orden $|G| \geq 2$.*

1. *There exists a squarefree zero sequence $S \in F(G)$ with $|S| = |G| - 1$.*
2. *Let $0 \neq g_0 \in G$ and $1 \leq k \leq \frac{|G|}{2} - 1$ with $k \neq 2$, if G is an elementary 2-group. Then there exist a squarefree zero sequence $S \in F(G)$ with $g_0 \nmid S$ and $|S| = k$.*

The following corollary is a consequence of the above lemma.

Corollary 1 *Let G is an elementary 2-group de orden $|G| \geq 3$ such that $0 \neq x \in G$ and $4 \leq k \leq \frac{|G|}{2} - 1$. Then there exist a k-set A of zero-sum in G such that $x \notin A$.*

3 Existence conditions of $BO(k, G)$ for general abelian groups

Let G be a finite. In the following two theorems, the values $r_2(G) = 0$ or $r_2(G) = 1$ are considered to give an existence condition in the order G to have a k-barycentric set, for each $3 \leq k \leq |G| - 3$. Notice that from Remark 1 the parity of $|G|$ is used and depends on $r_2(G) = 0$ or $r_2(G) = 1$. Observe that the fact $r_2(G) = 0$ means that for each $g \in G$ we have $-g \neq g$. The results provided in this section allow us to establish the existence of $BO(k, G)$ with $3 \leq k \leq |G| - 3$ for cyclic groups and elementary p-groups. A relationship between the Harborth $g(G)$ and the k-barycentric Olson $BO(k, G)$ constants is presented. From these relations, we give exact values of $BO(k, G)$ for some groups where $g(G)$ exists. Finally we identify some conditions on certain groups G in order to provide the exact values of $BO(|G| - 3, G)$.

Theorem 1 *Let G be a finite abelian group such that $r_2(G) = 0$ and $3 \leq k \leq |G| - 3$. Then $BO(k, G) \leq |G|$.*

Proof. Assuming $|G| \geq 9$. Let A be a zero-sum set of G such that $|A| = 3$ with $0 \notin A$ and we consider $B = \{-a : a \in A\}$. Notice that the sets $A \cup \{0\}$, $A \setminus \{a\} \cup B \setminus \{-a\}\} \cup \{0\}$ for some $a \in A$ and $A \cup B \cup \{0\}$ over G are k-barycentric, then $BO(k, G) \leq |G|$ for $k = 4$, 5 y 7.

Let $C = G \setminus (A \cup B \cup \{0\})$. Notice that since $|G| \geq 9$ and also odd then $|C| \geq 2$ is even. Moreover for all $c \in C$ we can see that $-c \in C$, assuming the contrary, we have a contradiction. Hence there exists $E \subseteq C$ with $2 \leq |E| \leq |C|$ conformed by elements a and its opposite. Since $|E|$ is even then $E \cup A \cup \{0\}$ or $E \cup A \cup B \cup \{0\}$ constitute the k-barycentric sets even or odd with barycenter 0, over G. Notice that $6 \leq k \leq |G| - 3$ with $k \neq 7$.

Moreover, since for all $0 \neq g \in G$ the set $\{g, -g, 0\}$ over G is a zero-sum then $BO(3, G) \leq |G|$.

Now, we consider the finite abelian groups G of order 3, 5 and 7. Observe that these groups are cyclic. In what follows we consider the existence of $BO(k, G)$. By Lemma 1 we have that $BO(3, C_3) = 3$, $BO(5, C_5) = 5$ and $BO(7, C_7) = 7$. Moreover by Proposition 1 we have that $BO(4, C_5)$ and $BO(6, C_7)$ does not exist and $BO(3, C_5) = 3$ and $BO(5, C_7) = 5$. Moreover, the 4-subset $A = \{0, 1, 2, 4\}$ over C_7 a zero-sum and $0 \in A$, in consequence $BO(4, C_7) \leq 7$ and for all $0 \neq a \in C_7$ the 3-subset $A = \{0, a, -a\}$ a zero-sum and $0 \in A$, hence $BO(3, C_7) \leq 7$. ■

The following two corollaries are a direct consequence of the above theorem.

Corollary 2 *Let C_n be a cyclic group such that $r_2(C_n) = 0$ and $3 \leq k \leq n - 3$. Then $BO(k, C_n) \leq n$.*

Corollary 3 *Let C_p^m be a elementary p-group such that $r_2(C_p^m) = 0$ and $3 \leq k \leq p^m - 3$. Then $BO(k, C_p^m) \leq p^m$.*

Theorem 2 *Let G be a finite abelian group such that $r_2(G) = 1$ and $3 \leq k \leq |G| - 3$ a positive integer. Then $BO(k, G) \leq |G|$.*

Proof. Assuming $|G| \geq 8$. Let $b^* \in G$ the only element of order 2. Let A be a 3-subset with zero-sum over G such that $b^* \in A$, $0 \notin A$ and $B = \{-a : a \in A\} \setminus \{b^*\}$. It is clear that the sets $A \cup \{0\}$ and $A \setminus \{b^*\} \cup B \cup \{0\}$ over G are barycentric with barycenter 0. Hence $BO(k, G) \leq |G|$ for $k = 4$ and 5.

Consider now, the set $C = G \setminus (A \cup B \cup \{0\})$. By Remark 1 G is even and then since $|A \cup B \cup \{0\}| = 6$ we have that $|C| \geq 2$ is even. Moreover for each $c \in C$ we have $-c \in C$, assuming the contrary we have a contradiction. Therefore there exists $E \subseteq C$ with zero-sum and $2 \leq |E| \leq |C|$ conformed by elements in C and its opposite. Hence the sets $E \cup A \cup \{0\}$ and $E \cup A \setminus \{b^*\} \cup B \cup \{0\}$ give the k-barycentric sets over G, k even and odd with barycenter 0 such that $6 \leq k \leq |G| - 3$.

Moreover, since for all $b^* \neq g \in G$ the set $\{g, -g, 0\}$ of G has zero-sum then $BO(3, G) \leq |G|$.

Now, we consider the finite abelian groups G of order 4 and 6. Observe that these groups are cyclic. In what follows we consider the existence of $BO(k, G)$. By Lemma 1 we have that $BO(4, C_4)$ and $BO(6, C_6)$ does not exist. Moreover by Proposition 1 we have that $BO(3, C_4) = 3$ and $BO(5, C_6) = 5$. Moreover, for all $3 \neq a \in C_6$ the 3-subset $A = \{0, a, -a\}$ a zero-sum and $0 \in A$, hence $BO(3, C_6) \leq 6$. ■

The following corollary is a consequence of the above theorem.

Corollary 4 *Let C_n be a cyclic group such that $r_2(C_n) = 1$ and $3 \leq k \leq n-3$. Then $BO(k, C_n) \leq n$.*

Theorem 3 *Let G be a finite abelian group with $|G| \geq 8$, $r_2(G) = 1$ and $3 \nmid |G|$. Then $BO(|G| - 3, G) = |G| - 2$.*

Proof. Let $b^* \in G$ be the only element with order 2. Let $A \subseteq G$ be such that $|A| = |G| - 2$. Assuming that $A = G \setminus \{a_1, a_2\}$ and consider $B = A \setminus [\{b^* + 2a_1 - a_2, b^* + 2a_2 - a_1\} \cup S_{a_1+a_2-b^*}]$. Since $|G| \geq 8$ then $|B| = |A| - 2 - |S_{a_1+a_2-b^*}| \geq (|G| - 2) - 2 - 2 = |G| - 6 > 0$. Hence $B \neq \emptyset$.

Let $b \in B \subseteq A$ be and consider the $(|G| - 3)$-subset $A \setminus \{b\}$ of A and we will see that $A \setminus \{b\}$ is a $(|G| - 3)$-barycentric set of A. We have that $\sigma(A \setminus \{b\}) = \sigma(A) - \sigma(b) = \sigma(G) - a_1 - a_2 - b = b^* - a_1 - a_2 - b$. Moreover, by Proposition 2 we have $-3 \cdot G = G$, then $\sigma(A \setminus \{b\}) = b^* - a_1 - a_2 - b = -3c$ for some $c \in G$. If $c = a_1$, then $b = b^* + 2a_1 - a_2 \notin B$, contradiction. If $c = a_2$, then $b = b^* + 2a_2 - a_1 \notin B$, contradiction. if $c = b$, then $2b = a_1 + a_2 - b^*$, in consequence $b \in S_{a_1+a_2-b^*} \nsubseteq B$, contradiction. Hence, $c \in G \setminus \{a_1, a_2, b\} = A \setminus \{b\}$ and therefore $\sigma(A \setminus \{b\}) = b^* - a_1 - a_2 - b = -3c = (|G| - 3)\, c$, for some $c \in A \setminus \{b\}$. Hence $A \setminus \{b\}$ is a $(|G|-3)$-barycentric set of A, i.e., $BO\,(|G|-3, G) \leq |G|-2$.

Now we see, $|G|-2 \leq BO\,(|G|-3, G)$. Consider the set $B = G \setminus [\{0, b^*\} \cup S_{b^*}]$. Since $|G| \geq 8$ then, $|B| = |G| - 2 - |S_{b^*}| \geq |G| - 2 - 2 = |G| - 4 > 0$. So that $B \neq \emptyset$.

Let $b \in B$ be, then $2b \neq b$ and $2b \neq b^*$ since if $2b = b^*$,$b \in S_{b^*}$. Consider $A = G \setminus \{b^*, b, 2b\}$, then $|A| = |G| - 3$ and $\sigma(A) = \sigma(G \setminus \{b^*, b, 2b\}) = \sigma(G) - b^* - b - 2b = b^* - b^* - b - 2b = -3b$. If $\sigma(A) = -3c$ for some $c \in A$, then $-3b = -3c$, in consequence $b = c$, this is a contradiction with the fact that $b \notin A$, that is to say, A it is not a $(|G|-3)$-barycentric set of G. So that $|G| - 2 \leq BO\,(|G|-3, G)$. Therefore, $BO\,(|G|-3, G) = |G| - 2$. ■

The following corollary is a consequence of the above theorem.

Corollary 5 *Let C_n be a cyclic group with $n \geq 8$, $r_2(C_n) = 1$ and $3 \nmid n$. Then $BO\,(n-3, G) = n-2$.*

Theorem 4 *Let $m \geq 2$ be then we have that $BO(3^m - 3, C_3^m) = 3^m - 2$.*

Proof. By Proposition 3 we have that $3^m - 2 \leq BO(3^m - 3, C_3^m)$. Let A be a $(k-2)$-subset over C_3^m. If $\sigma(A) \in A$, then the $(3^m - 3)$-subset $B = A \setminus \{\sigma(A)\}$ of A is a zero-sum. So that $\sigma(B) = 0 = (3^m - 3)b$ for each $b \in B$. Hence $B = A \setminus \{\sigma(A)\}$ is a $(3^m - 3)$-barycentric set.

Assuming that $\sigma(A) \notin A$, then $\sigma(A) \in A^c$ where A^c is a 2-subset over C_3^m. In consequence $A^c = \{\sigma(A), a\}$ with $\sigma(A) \neq a$. Since $\sigma(C_3^m) = 0$ and $\sigma(C_3^m) = \sigma(A) + \sigma(A^c)$, then $\sigma(A^c) = -\sigma(A) \Rightarrow \sigma(A) + a = -\sigma(A) \Rightarrow a + 2\sigma(A) = 0 = 3a \Rightarrow a = \sigma(A)$, a contradiction with the fact that $\sigma(A) \neq a$. Therefore, $BO(3^m - 3, C_3^m) = 3^m - 2$. ■

In what follows we consider the Harborth constant and we give its relationship with the k-barycentric Olson constant.

Definition 1 *Let G be a finite abelian group. The Harborth constant, denoted $g(G)$, is defined as the smallest positive integer ℓ such that each set $A \subseteq G$ with $|A| = \ell$ contains a subset B with $|B| = \exp(G)$ with zero-sum.*

The following remark and theorem establishes a relationship between the Harborth constant and the zero-sum problem.

Remark 2 *Kemnitz showed $g(C_p^2) = 2p-1$ for $p \in \{3,5,7\}$ in [9]. In particular, $g(C_3^2) = 5$. More recently Gao and Thangadurai [4] showed $g(C_p^2) = 2p-1$ for prime $p \geq 67$ and $g(C_4^2) = 9$. In [2] we can find other values for elementary 3-group; for example $g(C_3^3) = 10$, $g(C_3^3) = 21$, $g(C_3^5) = 46$ and $112 \leq g(C_3^6) \leq 114$ [1, 7, 8, 12].*

Theorem 5 *([11], Theorem 1.1)*

$$g(C_2 \oplus C_{2n}) = \begin{cases} 2n+2 & \text{if } n \text{ is even,} \\ 2n+3 & \text{if } n \text{ is odd.} \end{cases}$$

The following result determines the exact values of $BO(\exp(G), G)$ in finite abelian groups G where $g(G)$ there exists.

Theorem 6 *Let G be a finite abelian group where $g(G)$ exists. Then $BO(\exp(G), G) = g(G)$.*

Proof. Let $A \subseteq G$ be such that $|A| = g(G)$, then there exits $B \subseteq A$ with $|B| = \exp(G)$ such that $\sigma(B) = 0$. Therefore $\sigma(B) = 0 = \exp(G)b$ for all $b \in B$. Hence B is a $(\exp(G))$-barycentric subset of A, so that $BO(\exp(G), G) \leq g(G)$. Assuming that $A \subseteq G$ with $|A| = BO(\exp(G), G)$, then A contains a $(\exp(G))$-subset such that $\sigma(B) = (\exp(G))b = 0$ for all $b \in B$. So that B is a $(\exp(G))$-subset with zero-sum of A, that is to say $g(G) \leq BO(\exp(G), G)$. Therefore, $BO(\exp(G), G) = g(G)$. ■

The following result gives the exact values of $BO(\exp(G) + 1, G)$ for finite abelian groups where $g(G)$ exists and $g(G) \geq \exp(G) + 1$.

Theorem 7 *Let G be a finite abelian group such that $g(G)$ exists and $g(G) \geq \exp(G) + 1$. Then $BO(\exp(G) + 1, G) = g(G)$.*

Proof. Let $A \subseteq G$ be such that $|A| = g(G) \geq \exp(G) + 1$, then there exists $B \subseteq A$ with $|B| = \exp(G)$ such that $\sigma(B) = 0$. Now, since $|A| \geq |B| + 1$, then there exist some $a \in A \setminus B$. Let $C = B \cup \{a\}$ be then $|C| = \exp(G) + 1$ and we have that $\sigma(C) = \sigma(B) + \sigma(\{a\}) = 0 + a = a = 0 + a = \exp(G)a + a = (\exp(G) + 1)a$. Therefore C is a $(\exp(G) + 1)$-barycentric subset of A, hence, $BO(\exp(G) + 1, G) \leq g(G)$.

Assuming that $A \subseteq G$ such that $|A| = BO(\exp(G) + 1, G)$, hence there exists $B \subseteq A$ such that$|B| = \exp(G) + 1$, hence $\sigma(B) = (\exp(G) + 1)b$ with $b \in B$. Let $C = B \setminus b$ be a $(\exp(G))$-subset of A such that $\sigma(C) = \sigma(B) - \sigma\{b\} = (\exp(G) + 1)b - b = \exp(G)b + b - b = 0$. Therefore $C \subseteq A$ is a $(\exp(G))$-subset with a zero-sum, in consequence $g(G) \leq BO(\exp(G) + 1, G)$. Therefore, $BO(\exp(G) + 1, G) = g(G)$. ■

The following corollary is a consequence of Theorem 6 and Remark 2.

Corollary 6

$BO(3, C_3^2) = 5$, $BO(3, C_3^3) = 10$, $BO(3, C_3^4) = 21$,$BO(3, C_3^5) = 46$ and $112 \leq BO(3, C_3^6) \leq 114$.

The following corollary is a consequence of Theorem 7 and Remark 2.

Corollary 7

$BO(4, C_3^2) = 5$, $BO(4, C_3^3) = 10$, $BO(4, C_3^4) = 21$, $BO(4, C_3^5) = 46$ and $112 \leq BO(4, C_3^6) \leq 114$.

The following corollary is a consequence of Theorem 7 and Remark 2.

Corollary 8 $BO(2n, C_2 \oplus C_{2n}) = \begin{cases} 2n+2 & \textit{if } n \textit{ is even,} \\ 2n+3 & \textit{if } n \textit{ is odd.} \end{cases}$

The following corollary is a consequence of Theorem 7 and Theorem 5.

Corollary 9 $BO(2n+1, C_2 \oplus C_{2n}) = \begin{cases} 2n+2 & \textit{if } n \textit{ is even,} \\ 2n+3 & \textit{if } n \textit{ is odd.} \end{cases}$

Theorem 8 *([13], Theorem 4.2) Let $p \geq 7$ be an prime number and $\frac{p+1}{2} \leq k \leq p-3$. Then $BO(k, C_p) = k+1$.*

Theorem 9 *([13], Theorem 4.3) Let $p \geq 7$ be an integer prime number and $k = \frac{p-1}{2}$. Then*

$$BO(k, C_p) = \begin{cases} k+1 & \textit{if the multiplicity order of 2 module p is odd} \\ k+2 & \textit{if it is even.} \end{cases}$$

4 Existence conditions of $BO(k, G)$ for elementary 2-groups

Let C_2^m be an elementary 2-group of order 2^m. From the results cited in [13] we have that: $BO(2^m, C_2^m) = 2^m$, $BO(2^m - 1, C_2^m) = 2^m + 1$ and $BO(2^m - 2, C_2^m) = 2^m + 1$. In this section we study the existence of $BO(k, C_2^m)$ for $3 \leq k \leq 2^m - 3$. In some cases when $BO(k, C_2^m)$ exists, we give its exact value.

The following result is a consequence of Proposition 4 and Corollary 1.

Corollary 10 *Let k be an even integer such that $4 \leq k \leq 2^{m-1} - 1$. Then $BO(k, C_2^m) < 2^m$.*

The following result provides the existence of $BO(2^{m-1}, C_2^m)$.

Theorem 10 $BO(2^{m-1}, C_2^m) \leq 2^m$.

Proof. Assuming that $BO(2^{m-1}, C_2^m) = 2^m + 1$, i.e., each (2^{m-1})-subset A of C_2^m verifies $\sigma(A) \neq 0$ and $\sigma(A^c) \neq 0$. If $0 \in A$, then the $(2^{m-1} - 1)$-subset $B = A \setminus \{0\}$ verifies that $\sigma(B) \neq 0$, that is to say, there is no $(2^{m-1} - 1)$-subset B with zero-sum over C_2^m. Else $0 \in A^c$, then the $(2^{m-1} - 1)$-subset $C = A^c \setminus \{0\}$ verifies that $\sigma(C) \neq 0$; hence, there is no $(2^{m-1} - 1)$-subset C with zero-sum over C_2^m; then a contradiction with Lemma 2. Therefore, $BO(2^{m-1}, C_2^m) \leq 2^m$. ■

The following result gives the existence of $BO(k, C_2^m)$ for even k and $2^{m-1} + 2 \leq k \leq 2^m - 4$.

Theorem 11 *Let k be an even number such that* $2^{m-1} + 2 \leq k \leq 2^m - 4$. *Then* $BO(k, C_2^m) \leq 2^m$.

Proof. Assuming that $BO(k, C_2^m) = 2^m + 1$, i.e., each k-subset A over C_2^m is not barycentric, in consequence $\sigma(A) \neq 0$ and $\sigma(A^c) \neq 0$. Notice that $|A^c| = 2^m - k$ is an even integer and we have that $4 \leq 2^m - k \leq 2^{m-1} - 1$. So that C_2^m does not contain a $(2^m - k)$-subset A^c with zero-sum, therefore a contradiction with Corollary 10. So that, $BO(k, C_2^m) \leq 2^m$. ■

The following results follow from the last three results .

Corollary 11 *Let k be an even integer such that* $4 \leq k \leq 2^m - 4$. *Then* $BO(k, C_2^m) \leq 2^m$.

In order to complete the existence of $BO(k, C_2^m)$, we need to show that $BO(k, C_2^m) \leq 2^m$ for all even integers $3 < k \leq 2^m - 3$.

The following result shows the inexistence of $BO(3, C_2^m)$.

Theorem 12 $BO(3, C_2^m) = 2^m + 1$.

Proof. Assuming that $BO(3, C_2^m) \leq 2^m$, that is to say, there exists a 3-subset A in C_2^m such that $\sigma(A) \in A$. Let $B = A \setminus \{\sigma(A)\}$ be a 2-subset in C_2^m such that $\sigma(B) = 0 \in \sum_2 C_2^m$; therefore a contradiction with proposition 4.v. Hence, $BO(3, C_2^m) = 2^m + 1$. ■

The following result gives the exact values of $BO(2^m - 3, C_2^m)$.

Theorem 13 $BO(2^m - 3, C_2^m) = 2^m - 3$.

Proof. Assuming that $BO(2^m - 3, C_2^m) = 2^m + 1$, i.e., each $(2^m - 3)$-subset A over C_2^m is not barycentric, in consequence $\sigma(A) \notin A$; hence $\sigma(A) \in A^c$. Since $\sigma(A) = \sigma(A^c)$, then $\sigma(A^c) \in A^c$. So that, there exists a barycentric

3-subset A^c in C_2^m, that is to say, $BO(3, C_2^m) \leq 2^m$; hence a contradiction with the fact that $BO(3, C_2^m) = 2^m + 1$. Therefore, $BO(2^m - 3, C_2^m) = 2^m - 3$. ■

To finalize the discussion on the existence conditions of $BO(k, C_2^m)$ for the odd integers k in $5 \leq k \leq C_2^m - 5$ we will use the following results:

Proposition 5 *Let k be an even number and $BO(k, C_2^m) = q$. Then $q > k$.*

Proof. Assuming that $BO(k, C_2^m) = k$, i.e., for each k-subset A over C_2^m we have that $\sigma(A) = 0$. Let A be a k-barycentric set over C_2^m such that $0 \in A$ and consider the $(k-1)$-subset $B = A \setminus \{0\}$ over C_2^m, hence $\sigma(B) = \sigma(A) = 0$. Let $0 \neq c \in B^c$ be and consider the k-subset $D = B \cup \{c\}$ over C_2^m, hence $\sigma(D) = \sigma(B) + \sigma(\{c\}) = 0 + c = c \neq 0$. Therefore there exists a non barycentric k-subset D over C_2^m. Hence a contradiction with the fact that $BO(k, C_2^m) = k$. In consequence, $q > k$. ■

Theorem 14 *Let k be an even integer such that $4 \leq k \leq 2^m - 4$. If $BO(k, C_2^m) = q$, then $BO(k+1, C_2^m) = q$.*

Proof. Let A be a q-set over C_m^2 and B a k-subset over A such that $\sigma(B) = 0$. Since $|A| = q > k = |B|$, then there exists $a \in A \setminus B$. Let us consider the set $C = B \cup \{a\}$, notice that it is a $(k+1)$-subset over A such that $\sigma(C) = \sigma(B) + \sigma(\{a\}) = 0 + a = a$ with $a \in C$, that is to say, C is a barycentric $(k+1)$-subset in A. Hence $BO(k+1, C_2^m) \leq q$.

Assuming that A is a subset over C_m^2 such that $|A| = BO(k+1, C_2^m)$, then A contains a $(k+1)$-subset B such that $\sigma(B) = (k+1)b = kb + b = 0 + b = b$ for some $b \in B$. Let $C = B \setminus \{b\}$ be the k-subset of A such that $\sigma(C) = \sigma(B) - \sigma(\{b\}) = b - b = 0$. Hence C is a barycentric k-subset in A, i.e., $q \leq BO(k+1, C_2^m)$. Hence, $BO(k+1, C_2^m) = q$. ■

The following result is a direct consequence of the above theorem.

Corollary 12 *Let k be an odd integer such that $5 \leq k \leq 2^m - 5$. Then $BO(k, C_2^m) \leq 2^m$.*

The following result proves the increase of the values of $BO(k, C_2^m)$ when k is odd and $4 \leq k \leq 2^m - 3$.

Proposition 6 *Let k be an odd integer in $4 \leq k \leq 2^m - 5$. If $BO(k, C_2^m) = q_1$ and $BO(k+1, C_2^m) = q_2$, then $q_1 \leq q_2$.*

Proof. Assuming that $q_2 < q_1$. Let A be a q_2-set over C_2^m and B a barycentric $(k+1)$-subset of A, that is to say, $\sigma(B) = 0$. Let $b \in B$ be and consider the k-subset $C = B \setminus \{b\}$ so that $\sigma(C) = \sigma(B) - \sigma(\{b\}) = 0 - b = -b = b$, i.e., C is a barycentric k-subset of A. Hence $BO(k, C_2^m) \leq q_2 < q_1$, then a contradiction with the fact that $BO(k, C_2^m) = q_1$. Therefore, $q_1 \leq q_2$. ∎

The following result is a direct consequence of the above result and Theorem 14.

Corollary 13 *Let k be an integer such that $4 \leq k \leq 2^m - 4$. Then $BO(k+1, C_2^m) \geq BO(k, C_2^m)$.*

5 Conclusions

The goal of the present paper was to continue with the work in [13] for $3 \leq k \leq |G| - 3$. Our present main results are Theorem 1 and Theorem 2. The consequence of these two theorems were the complete existence conditions of cyclic groups and elementary p-groups. Moreover, in Section 4 the existence conditions for elementary 2-groups of our constant $BO(k, G)$ was completely determined. The problem of the existence of $BO(k, G)$ for all abelian groups G remains open, and also the problem of assigning exact values of the k-barycentric Olson constant when $BO(k, G)$ exists; some examples are the interesting results given in Theorem 3 and Theorem 4. The relation between the Harborth and the k-barycentric Olson constants established in this paper could be a good option to provide their exact values.

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