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HERMITE-HADAMARD INEQUALITIES TYPE FOR RAINA'S FRACTIONAL INTEGRAL OPERATOR USING $\eta-$CONVEX FUNCTIONS

DESIGUALDADES DE TIPO HERMITE-HADAMARD PARA EL OPERADOR INTEGRAL DE RAINA USANDO FUNCIONES $\eta-$CONVEXAS

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Abstract

In the present work, it is obtained some results concerning the integral inequality of Hermite-Hadamard, and others related to it, using $\eta-$convex functions and the fractional integral operator defined by R.K. Raina.

Keywords: Hermite-Hadamard inequality; $\eta-$convex functions; fractional integral operator.

Resumen

En el presente trabajo se encuentran resultados concernientes a la desigualdad integral de Hermite-Hadamard, y otras relacionadas con esta, usando funciones $\eta-$convexas y el operador integral fraccional definido por R.K. Raina.

Palabras clave: desigualdad de Hermite-Hadamard; funciones $\eta-$convexas; operadores integrales fraccionarios.

Mathematics Subject Classification: 26D10, 26A33, 26A51.

1 Introduction

It is well known that modern analysis, directly or indirectly, involves the applications of convexity. Due to its use and significant importance, the concept of convexity has been extended and generalized in several directions. The concept of convexity and its variant forms have played a fundamental role in the development of various fields. Convex functions are powerful tools for proving a large class of inequalities. They provide an elegant and unified treatment of the most important classical inequalities. In [5], M.E. Gordji et al. introduced the notion of $\eta-$convex functions as generalization of ordinary convex functions. There are many results associated with convex functions in the area of inequalities, one of them is the Hermite-Hadamard inequality (1), which occurs widely in the mathematical literature.

In [7], J. Hadamard stated his famous inequality in this way.

Theorem 1. *Let f be a convex function over $[a,b]$, $a<b$. If f is integrable over $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1}$$

Many others researchers have worked with this inequality using other generalizations of convexity, obtaining no less important results, as we can see in [2, 3, 4, 10, 11].

The inequalities involving more general fractional integral operators have also been considered in [1, 8, 12, 13]. Since work in this direction has gained much attention, we attempt to establish a general formulation in this article such that the essential facts covered by different fractional integrals become more clear and the implications yield certain new inequalities.

In particular, motivated by the work of M.A. Khan, Y. Khurshid and T. Ali in [9], this work contains some generalizations about the Hermite-Hadamard inequalities type in the context of $\eta-$ convex functions and fractional integral operator.

2 Preliminaries

This section contains some basic definitions, as well as some results that will be necessary for the development of the present work. The classical definition of convex function follows:

Definition 1. *Let $f : I \to R$ be a function defined over of the non-empty interval $I \subset R$. The function f is said to be convex on I if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for any $x, y \in I$ and $t \in [0, 1]$.

As a generalization of the definition of convexity, introduced in [5], we have the following.

Definition 2. *The function $f : [a, b] \to R$ is said to be $\eta-$convex function (or convex respect to η) on $[a, b]$ if the inequality*

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)),$$

holds for any $x, y \in I$ and $t \in [0, 1]$, and η is defined by $\eta : f([a, b]) \times f([a, b]) \to R$.

Note that when we choose $\eta(x,y) = x - y$ then we are dealing with the classic convex functions. The following is a example of this type of functions; this, and others can be found in [5]:

Example 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$f(x) = \begin{cases} -x & si\ x \geq 0, \\ x & si\ x < 0, \end{cases}$$

and define a bifunction η as $\eta(x,y) = -x - y$, for all $x, y \in (-\infty, 0]$. Then, f is a $\eta-$convex function but is not convex.

There are some important properties and results about $\eta-$convexity in [5] and [6].

Also in [5], the authors proved some important results but here we give only one of them in the following theorem based on the above definition, which is also known as $\eta-$convex version of the Hermite-Hadamard inequality.

Theorem 2. *Suppose that $f : [a,b] \to R$ is a $\eta-$convex function such that is bounded above on $f([a,b]) \times f([a,b])$. Then the following inequalities holds*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} &\leq \frac{1}{b-a}\int_a^b f(x)dx \qquad (2) \\ &\leq \frac{1}{2}\left(f(a)+f(b)\right) + \frac{1}{4}\left(\eta\left(f(a),f(b)\right) + \eta\left(f(b),f(a)\right)\right) \\ &\leq \frac{1}{2}\left(f(a)+f(b)\right) + \frac{M_\eta}{2}. \end{aligned}$$

Another important aspect for the development of this work is the following. In [12], R.K. Raina introduced a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k, \qquad (3)$$

where $\rho, \lambda > 0, |x| < R$, ($R$ is the set of real numbers), $\sigma = (\sigma(1), \ldots, \sigma(k), \ldots)$ is a bounded sequence of positive real numbers. Note that if we take in (3) $\rho = 1$, $\lambda = 0$ and $\sigma(k) = ((\alpha)_k(\beta)_k)/(\gamma)_k$ for $k = 0, 1, 2, \ldots$, where α, β and γ are parameters which can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, \ldots$), and the symbol $(a)_k$ denote the quantity

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\ldots(a+k-1), \qquad k = 0, 1, \ldots,$$

and restrict its domain to $|x| \leq 1$ (with $x \in \mathbb{C}$), then we have the classical Hypergeometric Function, that is

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = F(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!} x^k.$$

Using (3), in [1], R.P. Agarwal, M-J Luo, and R.K. Raina, defined the following left-sided and right-sided fractional integral operators respectively, as follows

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(x-t)^{\rho}\right]\varphi(t)dt, \quad (x > a), \tag{4}$$

and

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(t-x)^{\rho}\right]\varphi(t)dt, \quad (x < b), \tag{5}$$

where $\lambda, \rho > 0$, $w \in R$ and φ is such that the integral on the right side exists.

It is easy to verify that $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi$ are bounded integral operators on $L_p(a,b)$, $(1 \leq p \leq \infty)$, if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right] < \infty.$$

In fact, for $\varphi \in L_p\left((a,b)\right)$ we have

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right\|_p \leq \mathfrak{M}\left\|\varphi\right\|_p,$$

and

$$\left\|\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right\|_p \leq \mathfrak{M}\left\|\varphi\right\|_p,$$

where

$$\left\|\varphi\right\|_p = \left(\int_a^b |\varphi(x)|^p\, dx\right)^{1/p}.$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Here, we just point out that the classical Riemann-Liouville fractional integrals I^{α}_{a+} and I^{α}_{b-} of order α.

$$\left(I^{\alpha}_{a+}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\varphi(t)dt, \qquad (x > a, \alpha > 0),$$

and

$$\left(I^{\alpha}_{b-}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1}\varphi(t)dt, \qquad (x < b, \alpha > 0),$$

follow from (4) and (5) setting $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$.

The following Lemma is in [13, 14].

Lemma 1. *Let $\lambda, \rho > 0, w \in R$, and σ a sequence of non-negatives real numbers. Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$ and $\lambda > 0$. If $f' \in L_1([a,b])$ then the following equality for fractional integral operator holds*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)\right)\\
&= \frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left[\int_0^1 (1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}(1-t)^{\rho}]f'(ta+(1-t)b)dt\right.\\
&\left. - \int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}t^{\rho}]f'(ta+(1-t)b)dt\right].
\end{aligned}$$

3 Main results

Theorem 3. *Let $\lambda, \rho > 0, w \in R$, and σ a sequence of non-negatives real numbers. Let $f : [a,b] \to R$ be a $\eta-$convex function such that $\eta : f([a,b]) \times f([a,b]) \to R$ is upper bounded by M_η, then the following inequalities holds*

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right) - M_{\eta} \leq \frac{\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)\right)}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]} \qquad (6)\\
&\leq \frac{(f(a)+f(b))}{2} + \frac{(\eta(f(a),f(b)) + \eta(f(b),f(a)))}{2}\frac{\mathcal{F}^{\sigma_1}_{\rho,\lambda+2}[w(b-a)^{\rho}]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\\
&\leq \frac{(f(a)+f(b))}{2} + M_{\eta}\frac{\mathcal{F}^{\sigma_1}_{\rho,\lambda+2}[w(b-a)^{\rho}]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]},
\end{aligned}$$

where $\sigma_1(k) = \sigma(k)(k\rho+\lambda)$, for all $k = 0,1,2,\ldots$

Proof. From inequality (2) in Theorem 2, it is deduced that

$$f\left(\frac{x+y}{2}\right) - \frac{M_{\eta}}{2} \leq \frac{f(x)+f(y)}{2} + \frac{M_{\eta}}{2}. \qquad (7)$$

Let $x = ta+(1-t)b$ and $y = (1-t)a+tb$. Then (7) can be written as

$$f\left(\frac{a+b}{2}\right) - \frac{M_{\eta}}{2} \leq \frac{f(ta+(1-t)b) + f((1-t)a+tb)}{2} + \frac{M_{\eta}}{2},$$

and from this, is obtained that

$$2f\left(\frac{a+b}{2}\right)-M_\eta \leq f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)+M_\eta. \quad (8)$$

Multiplying by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right]$ in both sides of (8)

$$\begin{aligned}
&\left(2f\left(\frac{a+b}{2}\right)-M_\eta\right)t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right]\\
&\quad\leq\; t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right]f\left(ta+(1-t)b\right)\\
&\qquad+t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right]f\left((1-t)a+tb\right)+M_\eta t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right].
\end{aligned}$$

Integrating over $t\in[0,1]$

$$\begin{aligned}
&\left(2f\left(\frac{a+b}{2}\right)-M_\eta\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]\\
&\quad\leq\; \int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right]f\left(ta+(1-t)b\right)dt\\
&\qquad+\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right]f\left((1-t)a+tb\right)dt\\
&\qquad+M_\eta\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]. \qquad (9)
\end{aligned}$$

With a convenient change of variables it can be observed that

$$\begin{aligned}
&\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right]f\left(ta+(1-t)b\right)dt\\
&\qquad=\frac{1}{b-a}\int_a^b\left(\frac{b-x}{b-a}\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}\left(\frac{b-x}{b-a}\right)^{\rho}\right]f(x)dx\\
&\qquad=\frac{1}{(b-a)^{\lambda}}\int_a^b(b-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(b-x\right)^{\rho}\right]f(x)dx\\
&\qquad=\frac{1}{(b-a)^{\lambda}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right] f\left((1-t)a+tb\right)dt \\
&\qquad = \frac{1}{b-a}\int_a^b \left(\frac{x-a}{b-a}\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}\left(\frac{x-a}{b-a}\right)^{\rho}\right] f(x)dx \\
&\qquad = \frac{1}{(b-a)^{\lambda}}\int_a^b (x-a)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w\left(x-a\right)^{\rho}\right] f(x)dx \\
&\qquad = \frac{1}{(b-a)^{\lambda}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a).
\end{aligned}$$

Making the substitution of these values in (9), is obtained that

$$\begin{aligned}
&\left(2f\left(\frac{a+b}{2}\right)-M_{\eta}\right)\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right] \\
&\leq \frac{1}{(b-a)^{\lambda}}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)\right)+M_{\eta}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right],
\end{aligned}$$

that is,

$$f\left(\frac{a+b}{2}\right)-M_{\eta} \leq \frac{\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)\right)}{2\left(b-a\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}$$

and this is the left side of the inequality (6).

To find the right side of (6), the Definition 2 of $\eta-$convexity of f is used to establish the following inequalities:

$$\begin{aligned}
f\left(ta+(1-t)b\right) &\leq f(b)+t\eta\left(f(a),f(b)\right), \\
f\left((1-t)a+tb\right) &\leq f(a)+t\eta\left(f(b),f(a)\right).
\end{aligned}$$

Multiplying by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right]$ in both sides of each inequalities, is obtained

$$\begin{aligned}
&t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right] f\left(ta+(1-t)b\right) \\
&\quad \leq \left(f(b)+t\eta\left(f(a),f(b)\right)\right)t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right]
\end{aligned}$$

and

$$\begin{aligned}
&f\left((1-t)a+tb\right)t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right] \\
&\quad \leq \left(f(a)+t\eta\left(f(b),f(a)\right)\right)t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right].
\end{aligned}$$

Adding the inequalities and integrating over $t \in [0,1]$ one has to

$$\begin{aligned}
&\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right]\left(f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right)dt\\
&\qquad\leq \left(f(a)+f(b)\right)\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right]dt\\
&\qquad+\left(\eta\left(f(a),f(b)\right)+\eta\left(f(b),f(a)\right)\right)\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[w(b-a)^{\rho}t^{\rho}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)}{2\left(b-a\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\\
&\leq \frac{\left(f(a)+f(b)\right)}{2}+\frac{\left(\eta\left(f(a),f(b)\right)+\eta\left(f(b),f(a)\right)\right)}{2}\frac{\mathcal{F}^{\sigma_1}_{\rho,\lambda+2}\left[w(b-a)^{\rho}t^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]},
\end{aligned}$$

where

$$\sigma_1(k)=\sigma(k)(k\rho+\lambda).$$

The proof is complete. □

With an application of Theorem 3 the Theorem 2.1 in [9] can be obtained.

Corollary 3.1. *Let $f : [a,b] \to R$ be a $\eta-$convex function such that $\eta : f([a,b]) \times f([a,b]) \to R$ is upper bounded by M_η, then the following inequalities holds*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) - M_\eta &\leq \frac{\Gamma(\alpha+1)\left(\left(\mathcal{I}^\alpha_{a+}f\right)(b) + \left(\mathcal{I}^\alpha_{b-}f\right)(a)\right)}{2(b-a)^\alpha} \\ &\leq \frac{(f(a)+f(b))}{2} + \frac{\alpha\left(\eta\left(f(a),f(b)\right) + \eta\left(f(b),f(a)\right)\right)}{2(\alpha+1)} \\ &\leq \frac{(f(a)+f(b))}{2} + \frac{\alpha}{\alpha+1}M_\eta. \end{aligned}$$

Proof. Making $\lambda = \alpha,\ \sigma = (1,0,0,\ldots)$ and $w = 0$, it is gotten

$$\begin{aligned} \mathcal{F}^\sigma_{\rho,\lambda+1}\left[w(b-a)^\rho\right] &= \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)} = \frac{1}{\Gamma(\alpha+1)}, \\ \mathcal{F}^{\sigma_1}_{\rho,\lambda+2}\left[w(b-a)^\rho\right] &= \sum_{k=0}^{\infty} \frac{\sigma(k)(k\rho+\lambda)}{\Gamma(k\rho+\lambda+2)} = \frac{\alpha}{\Gamma(\alpha+2)}, \end{aligned}$$

and

$$\begin{aligned} \left(\mathcal{J}^\sigma_{\rho,\lambda,a+;w}f\right)(x) &= \left(\mathcal{I}^\alpha_{a+}f\right)(x), \\ \left(\mathcal{J}^\sigma_{\rho,\lambda,b-;w}f\right)(x) &= \left(\mathcal{I}^\alpha_{b-}f\right)(x). \end{aligned}$$

Making the substitution in (6) in Theorem 3, one has to

$$\begin{aligned} f\left(\frac{a+b}{2}\right) - M_\eta &\leq \frac{\Gamma(\alpha+1)\left(\left(\mathcal{I}^\alpha_{a+}f\right)(b) + \left(\mathcal{I}^\alpha_{b-}f\right)(a)\right)}{2(b-a)^\alpha} \\ &\leq \frac{(f(a)+f(b))}{2} + \frac{\alpha\left(\eta\left(f(a),f(b)\right) + \eta\left(f(b),f(a)\right)\right)}{2(\alpha+1)} \\ &\leq \frac{(f(a)+f(b))}{2} + \frac{\alpha}{(\alpha+1)}M_\eta. \end{aligned}$$

The proof is complete. □

Remark 1. *If $f : [a,b] \to R$ is a convex function, then the inequalities of Corollary (3.1) can be written as*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)\left(\left(\mathcal{I}^\alpha_{a+}f\right)(b) + \left(\mathcal{I}^\alpha_{b-}f\right)(a)\right)}{2(b-a)^\alpha} \leq \frac{(f(a)+f(b))}{2},$$

with the substitution $\eta(x,y) = x - y$, $\lambda = \alpha$, $\sigma = (1,0,0,\ldots)$ *and* $w = 0$. *This make a coincidence with Theorem 1.4 in [9], and [13]. Also, doing* $\alpha = 1$, *it is obtained*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)} \int_a^b f(x)dx \leq \frac{(f(a)+f(b))}{2}.$$

Theorem 4. *Let* $f : [a,b] \to R$ *be a differentiable function on* (a,b) *with* $a < b$. *If* $|f'|$ *is an* $\eta-$*convex function on* $[a,b]$, *then the following inequalities for fractional integral operator holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)\right)\right|$$
$$\leq \frac{(b-a)\,\mathcal{F}^{\sigma_1}_{\rho,\lambda+2}[w(b-a)^{\rho}]}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}]}\left(2\left|f'\right|(b) + \eta(\left|f'\right|(a), \left|f'\right|(b))\right),$$

where

$$\sigma_1(k) = \sigma(k)\left(1 - \left(\frac{1}{2}\right)^{k\rho+\lambda}\right),$$

for all $k = 0, 1, 2, \ldots$

Proof. Using the Lemma 1 and the $\eta-$convexity of $|f'|$, is obtained that

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)\right)\right|\\
&=\frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left|\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)}w^k(b-a)^{k\rho}\int_0^1(1-t)^{k\rho+\lambda}f'(ta+(1-t)b)dt\right.\\
&\qquad\left.-\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)}w^k(b-a)^{k\rho}\int_0^1 t^{k\rho+\lambda}f'(ta+(1-t)b)dt\right|\\
&=\frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left|\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)}w^k(b-a)^{k\rho}\times\right.\\
&\qquad\left.\left(\int_0^1(1-t)^{k\rho+\lambda}f'(ta+(1-t)b)dt-\int_0^1 t^{k\rho+\lambda}f'(ta+(1-t)b)dt\right)\right|\\
&\le\frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)}w^k(b-a)^{k\rho}\times\\
&\qquad\left(\int_0^1\left|(1-t)^{k\rho+\lambda}-t^{k\rho+\lambda}\right|f'(ta+(1-t)b)dt\right)\\
&\le\frac{b-a}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}w^k(b-a)^{k\rho}\times\\
&\qquad\left(\int_0^{1/2}\left((1-t)^{k\rho+\lambda}-t^{k\rho+\lambda}\right)\left(\left|f'\right|(b)+t\eta(\left|f'\right|(a),\left|f'\right|(b)\right)dt\right.\\
&\qquad\left.+\int_{1/2}^1\left(t^{k\rho+\lambda}-(1-t)^{k\rho+\lambda}\right)\left(\left|f'\right|(b)+t\eta(\left|f'\right|(a),\left|f'\right|(b))\right)dt\right)\\
&=\frac{b-a}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda+1)}w^k(b-a)^{k\rho}\times\\
&\qquad\left[\left|f'\right|(b)\left\{\int_0^{1/2}\left((1-t)^{k\rho+\lambda}-t^{k\rho+\lambda}\right)dt+\int_{1/2}^1\left(t^{k\rho+\lambda}-(1-t)^{k\rho+\lambda}\right)dt\right\}\right.\\
&\qquad+\eta(\left|f'\right|(a),\left|f'\right|(b))\left\{\int_0^{1/2}\left((1-t)^{k\rho+\lambda}-t^{k\rho+\lambda}\right)tdt\right.\\
&\qquad\left.\left.+\int_{1/2}^1\left(t^{k\rho+\lambda}-(1-t)^{k\rho+\lambda}\right)tdt\right\}\right].
\end{aligned}
$$

It is easy to verify that

$$
\begin{aligned}
\int_0^{1/2}\left((1-t)^{k\rho+\lambda}-t^{k\rho+\lambda}\right)dt &= \int_{1/2}^1\left(t^{k\rho+\lambda}-(1-t)^{k\rho+\lambda}\right)dt\\
&=\frac{1-\left(\frac{1}{2}\right)^{k\rho+\lambda}}{k\rho+\lambda+1},
\end{aligned}
$$

and

$$\begin{aligned}\int_0^{1/2}\left((1-t)^{k\rho+\lambda}-t^{k\rho+\lambda}\right)tdt &= \int_0^{1/2}\left(t(1-t)^{k\rho+\lambda}-t^{k\rho+\lambda+1}\right)dt\\ &= \int_{1/2}^{1}(1-u)\,u^{k\rho+\lambda}du\\ &= \frac{1-\left(\frac{1}{2}\right)^{k\rho+\lambda+1}}{k\rho+\lambda+1}-\frac{1}{k\rho+\lambda+2},\end{aligned}$$

$$\begin{aligned}\int_{1/2}^{1}\left(t^{k\rho+\lambda+1}-t(1-t)^{k\rho+\lambda}\right)dt &= \frac{1-\left(\frac{1}{2}\right)^{k\rho+\lambda+2}}{k\rho+\lambda+2}-\int_0^{1/2}(1-u)u^{k\rho+\lambda}\\ &= \frac{1}{k\rho+\lambda+2}-\frac{\left(\frac{1}{2}\right)^{k\rho+\lambda+1}}{k\rho+\lambda+1}.\end{aligned}$$

Therefore

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)\right)\right|\\ &\leq \frac{b-a}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma\left(\rho k+\lambda+1\right)}w^k(b-a)^{k\rho}\times\\ &\qquad\left(\left|f'\right|(b)\left(2\frac{1-\left(\frac{1}{2}\right)^{k\rho+\lambda}}{k\rho+\lambda+1}\right)+\eta(\left|f'\right|(a),\left|f'\right|(b))\left(\frac{1-\left(\frac{1}{2}\right)^{k\rho+\lambda}}{k\rho+\lambda+1}\right)\right),\end{aligned}$$

finally

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)\right)\right|\\ &\leq \frac{(b-a)\,\mathcal{F}^{\sigma_1}_{\rho,\lambda+2}\left[w(b-a)^{\rho}\right]}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left(2\left|f'\right|(b)+\eta(\left|f'\right|(a),\left|f'\right|(b))\right),\end{aligned}$$

where

$$\sigma_1(k)=\sigma(k)\left(1-\left(\frac{1}{2}\right)^{k\rho+\lambda}\right)$$

for all $k=0,1,2,\ldots$
The proof is complete. □

With an application of Theorem 4, the Theorem 3.2 in [9] is obtained.

Corollary 4.1. *Let $f:[a,b]\to R$ be a differentiable function on (a,b) with $a<b$. If $|f'|$ is an $\eta-$convex function on $[a,b]$, then the following inequalities for fractional integral holds*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\left(\mathcal{I}_{a+}^{\alpha}f\right)(b)+\left(\mathcal{I}_{b-}^{\alpha}f\right)(a)\right)\right|\\&\leq\frac{(b-a)}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(2\left|f'\right|(b)+\eta(\left|f'\right|(a),\left|f'\right|(b))\right).\end{aligned}$$

Proof. An application of Theorem 4 with the following substitution: $\lambda=\alpha$, $\sigma=(1,0,0,\ldots)$ and $w=0$, leads to

$$\mathcal{F}_{\rho,\lambda+1}^{\sigma}\left[w(b-a)^{\rho}\right]=\frac{1}{\Gamma(\alpha+1)},$$

$$\mathcal{F}_{\rho,\lambda+2}^{\sigma_1}\left[w(b-a)^{\rho}\right]=\frac{1-\left(\frac{1}{2}\right)^{\alpha}}{\Gamma(\alpha+2)},$$

also

$$\left(\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma}f\right)(x)=\left(\mathcal{I}_{a+}^{\alpha}f\right)(x),$$

$$\left(\mathcal{J}_{\rho,\lambda,b-;w}^{\sigma}f\right)(x)=\left(\mathcal{I}_{b-}^{\alpha}f\right)(x).$$

Making the substitution, is obtained that

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\left(\mathcal{I}_{a+}^{\alpha}f\right)(b)+\left(\mathcal{I}_{b-}^{\alpha}f\right)(a)\right)\right|\\&\leq\frac{(b-a)}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(2\left|f'\right|(b)+\eta(\left|f'\right|(a),\left|f'\right|(b))\right).\end{aligned}$$

The proof is complete. □

Remark 2. *If $f:[a,b]\to R$ is a convex function, then the inequalities of Corollary (4.1) can be written as*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\left(\mathcal{I}_{a+}^{\alpha}f\right)(b)+\left(\mathcal{I}_{b-}^{\alpha}f\right)(a)\right)\right|\\&\le\frac{(b-a)}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\left|f'\right|(a)+\left|f'\right|(b)\right),\end{aligned}$$

with the substitution $\eta(x,y)=x-y$, $\lambda=\alpha$, $\sigma=(1,0,0,\ldots)$ and $w=0$. This make a coincidence with Theorem 1.5 in [9], and [13]. Also, doing $\alpha=1$, it is gotten

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)}\int_a^b f(x)dx\right|\le\frac{(b-a)}{8}\left(\left|f'\right|(a)+\left|f'\right|(b)\right).$$

Theorem 5. *Let $f:[a,b]\to R$ be a differentiable function on (a,b) with $a<b$. If $|f'|^q$ is a $\eta-$convex function on $[a,b]$, with $q=p/(p+1)$ for some fixed $p>1$, then the following inequality for fractional integrals operator holds*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}_{\rho,\lambda+1}^{\sigma}\left[w(b-a)^{\rho}\right]}\left(\left(\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma}f\right)(b)+\left(\mathcal{J}_{\rho,\lambda,b-;w}^{\sigma}f\right)(a)\right)\right|\\&\le\frac{(b-a)\,\mathcal{F}_{\rho_1,\lambda_1+1}^{\sigma_1}\left[w(b-a)^{\rho_1/p}\right]}{\mathcal{F}_{\rho,\lambda+1}^{\sigma}\left[w(b-a)^{\rho}\right]}\times\\&\left[\left(\frac{|f'|^q(b)}{2}+\frac{\eta\left(|f'|^q(a),|f'|^q(b)\right)}{8}\right)^{1/q}+\left(\frac{|f'|^q(b)}{2}+\frac{3\eta\left(|f'|^q(a),|f'|^q(b)\right)}{8}\right)^{1/q}\right],\end{aligned}$$

where $\rho_1=\rho p$, $\lambda_1=\lambda p$ and

$$\sigma_1(k)=\sigma(k)\frac{2-\left(\frac{1}{2}\right)^{(k\rho+\lambda)p-1}}{(k\rho+\lambda)p+1},$$

for all $k=0,1,2,\ldots$

Proof. Using Lemma 1, the $\eta-$convexity of f and Hölder inequality

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)\right)\right|\\
&\leq \frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)}w^{k}(b-a)^{k\rho}\times\\
&\left[\left(\int_0^{1/2}\left((1-t)^{k\rho+\lambda}-t^{k\rho+\lambda}\right)^{p}dt\right)^{1/p}\left(\int_0^{1/2}\left(\left|f'\right|^{q}(b)+t\eta\left(\left|f'\right|^{q}(a),\left|f'\right|^{q}(b)\right)\right)dt\right)^{1/q}\right.\\
&\left.+\left(\int_{1/2}^{1}\left(t^{k\rho+\lambda}-(1-t)^{k\rho+\lambda}\right)^{p}dt\right)^{1/p}\left(\int_{1/2}^{1}\left(\left|f'\right|^{q}(b)+t\eta\left(\left|f'\right|^{q}(a),\left|f'\right|^{q}(b)\right)\right)dt\right)^{1/q}\right]\\
&\leq \frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)}w^{k}(b-a)^{k\rho}\times\\
&\left[\left(\int_0^{1/2}\left((1-t)^{(k\rho+\lambda)p}-t^{(k\rho+\lambda)p}\right)dt\right)^{1/p}\left(\int_0^{1/2}\left(\left|f'\right|^{q}(b)+t\eta\left(\left|f'\right|^{q}(a),\left|f'\right|^{q}(b)\right)\right)dt\right)^{1/q}\right.\\
&\left.+\left(\int_{1/2}^{1}\left(t^{(k\rho+\lambda)p}-(1-t)^{(k\rho+\lambda)p}\right)dt\right)^{1/p}\left(\int_{1/2}^{1}\left(\left|f'\right|^{q}(b)+t\eta\left(\left|f'\right|^{q}(a),\left|f'\right|^{q}(b)\right)\right)dt\right)^{1/q}\right]\\
&\leq \frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(k\rho+\lambda+1)}w^{k}(b-a)^{k\rho}\left(\frac{1-\left(\frac{1}{2}\right)^{(k\rho+\lambda)p+1}}{(k\rho+\lambda)\,p+1}\right)^{1/p}\\
&\left(\frac{\left|f'\right|^{q}(b)}{2}+\frac{\eta\left(\left|f'\right|^{q}(a),\left|f'\right|^{q}(b)\right)}{8}\right)^{1/q}+\left(\frac{\left|f'\right|^{q}(b)}{2}+\frac{3\eta\left(\left|f'\right|^{q}(a),\left|f'\right|^{q}(b)\right)}{8}\right)^{1/q}\\
&=\frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\sum_{k=0}^{\infty}\frac{\sigma_1(k)}{\Gamma(k\rho_1+\lambda_1+1)}w^{k}\left[(b-a)^{1/p}\right]^{k\rho_1}\times\\
&\left[\left(\frac{\left|f'\right|^{q}(b)}{2}+\frac{\eta\left(\left|f'\right|^{q}(a),\left|f'\right|^{q}(b)\right)}{8}\right)^{1/q}+\left(\frac{\left|f'\right|^{q}(b)}{2}+\frac{3\eta\left(\left|f'\right|^{q}(a),\left|f'\right|^{q}(b)\right)}{8}\right)^{1/q}\right],
\end{aligned}
$$

where

$$\rho_1=\rho p,\ \lambda_1=\lambda,$$

and

$$\sigma_1(k)=\sigma(k)\left(\frac{1-\left(\frac{1}{2}\right)^{(k\rho+\lambda)p}}{(k\rho+\lambda)\,p+1}\right)^{1/p},$$

for all $k=0,1,2,3,\ldots$

Therefore,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\left(\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f\right)(a)\right)\right|$$
$$\leq \frac{(b-a)\,\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[w(b-a)^{\rho_1/p}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\times$$
$$\left[\left(\frac{|f'|^q(b)}{2} + \frac{\eta\left(|f'|^q(a),|f'|^q(b)\right)}{8}\right)^{1/q} + \left(\frac{|f'|^q(b)}{2} + \frac{3\eta\left(|f'|^q(a),|f'|^q(b)\right)}{8}\right)^{1/q}\right].$$

The proof is complete. □

Corollary 5.1. *Let $f:[a,b]\to R$ be a differentiable function on (a,b) with $a<b$. If $|f'|^q$ is a $\eta-$convex function on $[a,b]$, with $q=p/(p+1)$ for some fixed $p>1$, then the following inequality for Riemann Liouville fractional integral holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\left(\mathcal{I}^{\alpha}_{a+}f\right)(b) + \left(\mathcal{I}^{\alpha}_{b-}f\right)(a)\right)\right|$$
$$\leq \frac{(b-a)\,\Gamma(\alpha+1)}{2\Gamma(\alpha p+1)}\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1}\right)^{1/p}\times$$
$$\left[\left(\frac{|f'|^q(b)}{2} + \frac{\eta\left(|f'|^q(a),|f'|^q(b)\right)}{8}\right)^{1/q} + \left(\frac{|f'|^q(b)}{2} + \frac{3\eta\left(|f'|^q(a),|f'|^q(b)\right)}{8}\right)^{1/q}\right].$$

Proof. Making $\lambda=\alpha,\ \sigma=(1,0,0,\ldots)$, and $w=0$ in Theorem 5 is obtained that

$$\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right] = \frac{1}{\Gamma(\alpha+1)},$$
$$\mathcal{F}^{\sigma_1}_{\rho_1,\lambda_1+1}\left[w(b-a)^{\rho_1/p}\right] = \frac{1}{\Gamma(\alpha p+1)}\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1}\right)^{1/p},$$

therefore

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\left(\mathcal{I}^{\alpha}_{a+}f\right)(b) + \left(\mathcal{I}^{\alpha}_{b-}f\right)(a)\right)\right|$$
$$\leq \frac{(b-a)\,\Gamma(\alpha+1)}{2\Gamma(\alpha p+1)}\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1}\right)^{1/p}\times$$
$$\left[\left(\frac{|f'|^q(b)}{2} + \frac{\eta\left(|f'|^q(a),|f'|^q(b)\right)}{8}\right)^{1/q} + \left(\frac{|f'|^q(b)}{2} + \frac{3\eta\left(|f'|^q(a),|f'|^q(b)\right)}{8}\right)^{1/q}\right].$$

The proof is complete. □

Remark 3. *Doing $\eta(x,y) = x - y$ in Corollary 5.1, is obtained that*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(\left(\mathcal{I}_{a+}^{\alpha} f\right)(b) + \left(\mathcal{I}_{b-}^{\alpha} f\right)(a)\right)\right|$$
$$\leq \frac{\Gamma(\alpha+1)(b-a)}{\Gamma(\alpha p+1)}\left(\frac{1-\left(\frac{1}{2}\right)^{\alpha p}}{\alpha p+1}\right)^{1/p} \times$$
$$\left(\left(\frac{|f'|^q(a)}{8} + \frac{3|f'|^q(b)}{8}\right)^{1/q} + \left(\frac{3|f'|^q(a)}{8} + \frac{|f'|^q(b)}{8}\right)^{1/q}\right).$$

This inequality corresponds for $|f'|^q$ as a convex function. In addition, doing $\alpha = 1$ one has to

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{2(b-a)}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)}{\Gamma(p+1)}\left(\frac{1-\left(\frac{1}{2}\right)^{p}}{p+1}\right)^{1/p} \times$$
$$\left(\left(\frac{|f'|^q(a)}{8} + \frac{3|f'|^q(b)}{8}\right)^{1/q} + \left(\frac{3|f'|^q(a)}{8} + \frac{|f'|^q(b)}{8}\right)^{1/q}\right).$$

4 Concluding remarks

In the development of the present work have been established some results that generalize, from the definition of Raina integral fractional operator and the use of $\eta-$convex functions, others previously found for the Riemann-Liouville fractional integral. In particular, those concerning the integral inequality of Hermite Hadamard.

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