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CHARACTERIZATION OF BMO USING WAVELETS THROUGH TRIEBEL-LIZORKIN SPACES

CARACTERIZACIÓN DE BMO USANDO ONDÍCULAS POR MEDIO DEL ESPACIO DE TRIEBEL-LIZORKIN

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Abstract

In the present article it is presented a characterization of all those functions in the space of bounded mean oscillation functions, BMO, in terms of an appropriate wavelet, using an isomorphism between the aforementioned space and the homogeneous space of Triebel-Lizorkin $\dot{F}_{\infty}^{0,2}$. In addition, a new inequality that involves the vector inequality of the maximal function of Hardy-Littlewood is proved.

Keywords: BMO function space; Triebel-Lizorkin's homogeneous space; wavelets.

Resumen

En el presente artículo se presenta una caracterización de todas aquellas funciones pertenecientes al espacio de oscilación media acotada, BMO, en términos de una apropiada ondícula, usando un isomorfismo entre el mencionado espacio de funciones y el espacio homogéneo de Triebel-Lizorkin $\dot{F}_{\infty}^{0,2}$. Además, se prueba una versión nueva que involucra la desigualdad vectorial de la función maximal de Hardy-Littlewood.

Palabras clave: espacio de funciones de oscilación media acotada; espacio homogéneo de Triebel-Lizorkin; ondículas.

Mathematics Subject Classification: Primary 22E46, 53C35, Secondary 57S20.

1 Introduction

Wavelets were introduced in early's 80, they have been of interest for the mathematical scientific community and other disciplines. The wavelet analysis has been used as an alternative for the windowed Fourier analysis, that is, for the case in which the objective is to measure the frequency content of a signal, while the case of wavelets is to compare several sizes of this signal with different resolutions. To fix a clear definition: a function $\psi \in L_2(\mathbb{R})$ is an orthonormal wavelet provided the system $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an orthonormal basis for $L_2(\mathbb{R})$ where

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k), \quad \text{for all } \ j,k \in \mathbb{Z}.$$

Besides providing us with orthogonal bases for the Hilbert space $L_2(\mathbb{R})$, some wavelets gives us natural basis for other topological spaces.

Recently, in 2008, Triebel [13] presented some results that associate the use of wavelets with the Besov function spaces and certain estimates for local means. Also, in 2010, Han and Lu [7] made use of the Littlewood-Paley theory and the multiparameter Hardy space theory.

Using the Littlewood-Paley theory, and the wavelet representation of functions in a fixed function space, it is possible to obtain characterization of such function space. To cite a few instances, in [8] it is the characterization of the functions in the Lebesgue space $L_p(\mathbb{R})$, $(1<p<\infty)$, Hardy space $H_1(\mathbb{R})$ and Sobolev space $L^{p,s}(\mathbb{R}),\ 1<p<\infty,\ s=1,2,3,\ldots$ Below it is showed these characterizations.

Theorem 1. *Let ψ be an orthonormal wavelet such that $\psi\in\mathcal{R}^0$. If $1<p<\infty$, there exist constants A_p and B_p, $0<A_p<B_p<\infty$ such that*

$$A_p\left\|f\right\|_{L_p(\mathbb{R})}\le\left\|\left(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\langle f,\psi_{j,k}\rangle|^2\,2^{\frac{j}{2}}\chi_{[2^{-j}k,2^{-j}(k+1)]}(\cdot)\right)^{1/2}\right\|_{L_p(\mathbb{R})}$$
$$\le B_p\left\|f\right\|_{L_p(\mathbb{R})},$$

for all $f\in L_p(\mathbb{R})$.

Theorem 2. *Let ψ be an orthonormal wavelet such that $\psi\in\mathcal{R}^0$. If $1<p<\infty$, there exist constants A_p and B_p, $0<A_p<B_p<\infty$ such that*

$$A_p\left\|f\right\|_{H_1(\mathbb{R})}\le\left\|\left(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\langle f,\psi_{j,k}\rangle|^2\,2^{\frac{j}{2}}\chi_{[2^{-j}k,2^{-j}(k+1)]}(\cdot)\right)^{1/2}\right\|_{L_p(\mathbb{R})}$$
$$\le B_p\left\|f\right\|_{H_1(\mathbb{R})},$$

for all $f\in H_1(\mathbb{R})$.

Theorem 3. *Let ψ in the Schwartz class, S, be a limited band orthonormal wavelet. For $1<p<\infty$ and $s=1,2,3,\ldots$; there exist constants $A_{p,s}$ and $B_{p,s}$, $0<A_{p_s}<B_{p,s}<\infty$ such that*

$$A_p\left\|f\right\|_{L^{p,s}(\mathbb{R})}\le\left\|\left(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\langle f,\psi_{j,k}\rangle|^2\left(1+2^{js}\right)2^{j}\chi_{[2^{-j}k,2^{-j}(k+1)]}(\cdot)\right)^{1/2}\right\|_{L_p(\mathbb{R})}$$
$$\le B_p\left\|f\right\|_{L^{p,s}(\mathbb{R})},$$

for all $f\in L^{p,s}(\mathbb{R})$.

Frazier, Jawerth and Weiss showed, in [5], that the homogeneous space of Triebel-Lizorkin $\dot{F}_1^{0,2}$ is isomorphic to the Hardy space H^1; also, Fefferman in [1], proved that BMO space is the dual space of H^1; and Frazier and Jawerth in [3], proved that $\dot{F}_\infty^{0,2}$ is the dual space of $\dot{F}_1^{0,2}$. We use these results to achieve a characterization of BMO, with wavelets coefficients, through functions in $\dot{F}_\infty^{0,2}$.

2 Preliminares

By $L_p(\mathbb{R}), 1 \leq p \leq \infty$ we denote the Banach space of equivalence classes of measurable functions on $\mathbb{R}$ whose p'th power is integrable (respectively, which are essentially bounded if $p = \infty$). The norm in $L_p(\mathbb{R})$ is defined by

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p}, \ (1 \leq p < \infty),$$

in the case of $p = \infty$

$$\|f\|_\infty = ess \sup |f(x)| \, .$$

Similarly, by $l_p(\mathbb{R}), 1 < p < \infty$ we denote the Banach space of sequences $\{a_k\}_{k\in\mathbb{Z}}$ of real (complex) numbers whose p'th power is summable (respectively, which are bounded if $p = \infty$). The norm in $l_p(\mathbb{R})$ is defined by

$$\left\|\{a_k\}_{k\in\mathbb{Z}}\right\|_p = \left(\sum_{k\in\mathbb{Z}} |a_k|^p \, dx \right)^{1/p},$$

in the case of $p = \infty$

$$\left\|\{a_k\}_{k\in\mathbb{Z}}\right\|_\infty = \sup_{k\in\mathbb{Z}} |a_k| \, .$$

When $p = 2$ the inner product of functions f, g in $L_2(\mathbb{R})$ is defined by

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx,$$

so that we shall say that two functions f, g are orthonormal if

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx = 0.$$

We will say that the sequence $\{f_n\}_{n\in\mathbb{Z}}$ is orthogonal if $\langle f_n, f_m \rangle = \delta_{n,m}$, where δ is the Dirac delta function

$$\delta_{n,m} = \begin{cases} 1, & \text{if } \ n = m, \\ 0, & \text{if } \ n \neq m. \end{cases}$$

The convolution $f * g$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}} f(t)g(x-t)dt.$$

The Schwartz class S, consist of those functions φ satisfying

$$i)\ \varphi \in C_{\infty}(\mathbb{R}),$$
$$ii)\ \sup_{m\leq k}\sup_{x\in\mathbb{R}}\left(1+x^2\right)^k\left|\varphi^{(m)}(x)\right|<\infty.$$

This kind of functions are known by rapidly decreasing functions.

We shall say that a distribution φ is a tempered distribution if $\varphi \in S^*$, where S^* is the dual space of S. Also, a tempered distribution φ is modulus polynomial if $\varphi \in S^*/P$ where P is the set of polynomials over $\mathbb{R}$. All these basic concepts can be found in [11].

The following lemma shows how a tempered distribution can be written in series using the Schwartz class of functions S, which have Fourier transform with compact support, and its proof is outlined in [8].

Lemma 1. *Let $\psi \in L^2(\mathbb{R})$ be a function such that supp$(\hat{\psi}) \subseteq [-\pi,\pi]$, and*

$$\sum_{j\in\mathbb{Z}}\left|\hat{\psi}(2^j\xi)\right|^2=1,\ \textit{for a.e.}\quad \xi\in\mathbb{R}-\{0\}\,.$$

If f is a tempered distribution then

$$f=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\langle f,\psi_{j,k}\rangle\,\psi_{j,k}.$$

The following result is mentioned in several publications, it will be useful for the present work, and next a proof of the same one is presented.

Proposition 1. *If f is a tempered distribution and $\varphi \in S$ then*

$$|\langle f, \varphi_{j,k}\rangle| \leq 2^{\frac{-j}{2}} \sup_{y \in I_{j,k}} |(\widetilde{\varphi}_{2^{-j}} * f)(y)|, \quad \textit{for all } j,k \in \mathbb{Z},$$

where

$$\varphi_{j,k}(x) = 2^{\frac{j}{2}} \varphi\left(2^j x - k\right), \ \ \widetilde{\varphi}(x) = \overline{\varphi(-x)}, \ \ \widetilde{\varphi}_{2^{-j}}(x) = \tfrac{1}{2^{-j}} \overline{\varphi(\tfrac{-x}{2^{-j}})}$$

and

$$I_{j,k} = [2^{-j}k, 2^{-j}(k+1)].$$

Proof. Let $\varphi \in S$. Since $\left|\overline{\varphi}^{(m)}(x)\right| = \left|\varphi^{(m)}(x)\right|$ we have that $\overline{\varphi} \in S$. In other hand, if f is tempered distribution, we can observe that

$$\langle f, \varphi\rangle = \int_{\mathbb{R}} f(t)\overline{\varphi}(t)dt = f(\overline{\varphi}),$$

so, treating f as a bounded linear operator, we get that $\langle f, \varphi\rangle$ is finite. Also we get

$$\begin{aligned}
\langle f, \varphi_{j,k}\rangle &= \int_{\mathbb{R}} f(t) \cdot \overline{\varphi_{j,k}}\,(t)\,dt \\
&= \int_{\mathbb{R}} f(t) \cdot \overline{2^{\frac{j}{2}}\,\varphi(2^j t - k)}\,dt \\
&= 2^{\frac{-j}{2}} \int_{\mathbb{R}} f(t) \cdot \frac{2^j}{2^j}\,\overline{\varphi(-2^j(-t + 2^{-j}k))}\,dt \\
&= 2^{\frac{-j}{2}} \int_{\mathbb{R}} f(t) \cdot \ \varphi_{2^{-j}}(2^{-j}k - t)dt \\
&= 2^{\frac{-j}{2}}\,(\widetilde{\varphi}_{2^{-j}} * f)\,(2^j k),
\end{aligned}$$

for all $j,k \in \mathbb{Z}$. In consequence, if $I_{j,k} = \left[2^{-j}k, 2^{-j}(k+1)\right]$ then

$$|\langle f, \varphi_{j,k}\rangle| \leq 2^{-j/2} \sup_{y \in I_{j,k}} |(\widetilde{\varphi}_{2^{-j}} * f)\,(y)|\,.$$

□

Given an orthonormal system $\{f_n\}_{n\in\mathbb{Z}}$ and a function f, we define the Fourier coefficients respect to the system $\{f_n\}_{n\in\mathbb{Z}}$ as

$$c_k = \langle f, f_k \rangle\,,\ k \in \mathbb{Z},$$

and the Fourier transform of a function $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}dx,$$

and the inverse transform by

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)e^{i\xi x}d\xi,$$

in the case of a tempered distribution f, the Fourier transform is also a tempered distribution defined by

$$\widehat{f}(\varphi) = f\left(\widehat{\varphi}\right),\ (\varphi \in S),$$

moreover

$$(f * \varphi)\,\hat{}\, = \widehat{f}\widehat{\varphi}.$$

It is known the following result showed by Hernandez and Weis in [8].

Lemma 2. *Let $\varepsilon > 0$. Let f, g be functions and $C_1, C_2 > 0$ such that*

$$|f(x)| \leq \frac{C_1}{(1+|x|)^{1+\varepsilon}} \ \ \textit{and} \ \ \ |g(x)| \leq \frac{C_2}{(1+|x|)^{1+\varepsilon}},$$

for all $x \in \mathbb{R}$. Then there exist $C > 0$ such that for all $j, k, l, m \in \mathbb{Z},\ j \geq l$ we have

$$|(f_{j,k} * g_{l,m})(x)| \leq \frac{C2^{\frac{j-1}{2}}}{\left(1+2^j\left|x-2^{-j}k-2^{-l}m\right|\right)^{1+\varepsilon}},$$

for all $x \in \mathbb{R}$.

We shall say that $\psi \in L_2(\mathbb{R})$ is a wavelet if $\{\psi_{j,k}\}_{i,k\in\mathbb{Z}}$ is an orthonormal basis for $L_2(\mathbb{R})$ where $\psi_{j,k}(x) = 2^{\frac{-j}{2}}\psi(2^jx-k)$.

The next Theorem gives us a characterization of the wavelets in terms of properties of their Fourier transform (see [8]).

Theorem 4. *A function $\psi \in L_2(\mathbb{R})$, with $\|\psi\|_{L_2(\mathbb{R})} = 1$, is an orthonormal wavelet if and only if*

$$\sum_{j\in\mathbb{Z}} \left|\widehat{\psi}\left(2^j\xi\right)\right|^2 = 1, \text{ for a.e. } \xi \in \mathbb{R},$$

and

$$\sum_{j=0}^{\infty} \widehat{\psi}\left(2^j\xi\right) \overline{\widehat{\psi}\left(2^j\left(\xi + 2m\pi\right)\right)} = 0, \text{ for a.e. } \xi \in \mathbb{R}, m \in 2\mathbb{Z}+1.$$

Definition 1. *A wavelet ψ is called limited band wavelet if supp$(\hat{\psi}) \subseteq I$ for some $I \subset \mathbb{R}$ with $l(I) < \infty$.*

Remark 1. *The Lemarié-Meyer wavelet is in the Schwartz class with Fourier transform given by*

$$\hat{\psi}(\xi) = b(\xi)e^{i\,\frac{\xi}{2}},$$

where

$$b(\xi) = \begin{cases} \sin\left(\frac{3}{4}\left(|\xi| - \frac{2}{3}\pi\right)\right) & \text{if } \frac{2}{3}\pi < |\xi| \leq \frac{4}{3}\pi, \\ \sin\left(\frac{3}{8}\left(\frac{8}{3}\pi - |\xi|\right)\right) & \text{if } \frac{4}{3}\pi < |\xi| \leq \frac{8}{4}\pi, \\ 0 & \text{in other case.} \end{cases}$$

2.1 Maximals functions

Let f be a locally integrable function over $\mathbb{R}$, we define the maximal function of Hardy-Littlewood by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{2r} \int_{|y-x|\leq r} |f(y)|\, dy.$$

An important result regarding this maximal function, and related to $L_p(\mathbb{R})$ functions, was established by Fefferman and Stein in [2].

Theorem 5. *Let $1 < p,\ q < \infty$, then there exist $C_{p,q}$ such that*

$$\left\|\left\{\sum_{i=1} (\mathcal{M}f_i)^q\right\}^{1/q}\right\|_{L^p(\mathbb{R})} \leq C_{p,q} \left\|\left\{\sum_{i=1} |f_i|^q\right\}^{1/q}\right\|_{L^p(\mathbb{R})},$$

for any sequence $f_i : i = 1, 2, 3, \ldots;$ of locally integrable functions.

As an immediately consequence, if f is a locally integrable function then the functions $f_1 = f, 0 = f_2 = f_3 = \ldots$; are locally integrable functions, and an application of Theorem 5 with $q = p$ shows that

$$\|\mathcal{M}(f)\|_p \leq C_p \|f\|_p \,, \quad p > 1.$$

Therefore, for any sequence $f_i : i = 1, 2, 3, \ldots$; of locally integrable functions we have

$$\|\mathcal{M}(f_i)\|_p \leq C_p \|f_i\|_p \,, \quad p > 1, \quad i = 1, 2, 3, \ldots \tag{1}$$

In the main results section it will be established a new similar result.

The following result is proved by Hernández and Weiss in [8].

Lemma 3. *Given $\varepsilon > 0$ and $1 \leq r < 1 + \varepsilon$ there exist C such that for any sequence $\{s_{j,k} : j, k \in \mathbb{Z}\}$ of complex numbers and any $x \in I_{j,k}$ we have*

(a) if $l \leq j$,

$$\sum_{m \in \mathbb{Z}} \frac{|s_{l,m}|}{\left(1 + 2^l \left|2^{-j}k - 2^{-l}m\right|\right)^{1+\varepsilon}} \leq C \left[\mathcal{M}\left(\sum_{m \in \mathbb{Z}} |s_{l,m}|^{1/r} \chi_{I_{l,m}}\right)(x)\right]^r,$$

(b) if $l \geq j$,

$$\sum_{m \in \mathbb{Z}} \frac{|s_{l,m}|}{\left(1 + 2^l \left|2^{-l}k - 2^{-j}m\right|\right)^{1+\varepsilon}} \leq 2^{(l-j)r} C \left[\mathcal{M}\left(\sum_{m \in \mathbb{Z}} |s_{l,m}|^{1/r} \chi_{I_{l,m}}\right)(x)\right]^r,$$

where $\mathcal{M}$ is the maximal function of Hardy-Littlewood.

For a function g on $\mathbb{R}$ and for a real number $\lambda > 0$, we consider the maximal function

$$g_\lambda^*(x) = \sup_{y \in \mathbb{R}} \frac{g\left(|x - y|\right)}{\left(1 + |y|\right)^\lambda}, \;\; x \in \mathbb{R},$$

Hernández and Weiss show a relation between these maximal functions in [8].

Lemma 4. *Let g be a limited band function defined over $\mathbb{R}$, such that $g_\lambda^*(x) < \infty$, for all $x \in \mathbb{R}$ and $\lambda > 0$. Then there exists a constant $C_\lambda > 0$ such that*

$$g_\lambda^*(x) \leq C_\lambda \left[\mathcal{M}\left(|g|^{1/\lambda}\right)(x)\right]^\lambda, \; x \in \mathbb{R}.$$

Also, the same authors in [8], proved the following Lemma.

Lemma 5. *Let φ be a limited band function, $f \in S'$, $0 < p \leq \infty$ such that $\varphi_{2^{-j}} * f \in L_p(\mathbb{R})$, for all $j \in \mathbb{Z}$. Then, for all $\lambda > 0$ there exists a constant C_λ such that*

$$\left(\varphi^{**}_{j,\lambda} f\right)(x) \leq C_\lambda \left[\mathcal{M}\left(|\varphi_{2^{-j}} * f|^{1/\lambda}\right)(x)\right]^\lambda, \quad x \in \mathbb{R},$$

where

$$\left(\varphi^{**}_{j,\lambda} f\right)(x) = \sup_{y\in\mathbb{R}} \frac{|(\varphi_{2^{-j}} * f)(x-y)|}{(1 + 2^j |y|)^\lambda}, \;\; x \in \mathbb{R},$$

and $\varphi_t(x) = (1/t)\varphi(x/t)$.

2.2 Hardy, BMO and Triebel-Lizorkin function spaces

Garcia and Rubio de Francia in [6] wrote about the following function space. Given the functions

$$P_t(x) = \frac{1}{\pi}\frac{t}{x^2+t^2},$$

$$Q_t(x) = \frac{1}{\pi}\frac{x}{x^2+t^2},$$

and a function $f \in L_p(\mathbb{R}),\ 1 \leq p < \infty$, we define the functions

$$u(x,t) = (P_t * f)(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{t}{(x-y)^2+t^2} f(y)dy,$$

$$v(x,t) = (Q_t * f)(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x-y}{(x-y)^2+t^2} f(y)dy,$$

both are harmonic functions and

$$\lim_{t\to 0^+} u(x,t) = f(x) \text{ a.e in } \mathbb{R}.$$

We define $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = u(x,t) + iv(x,t) = \frac{i}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y} dy,$$

then F is an analytic function.

The set

$$Re(H)^1 = \left\{ f \in L^1(\mathbb{R}) : \sup_{t>0} \left(\int_{\mathbb{R}} |F(x+it)|\, dx \right) < \infty \right\},$$

equipped with

$$\|f\| = \sup_{t>0} \left(\int_{\mathbb{R}} |F(x+it)|\, dx \right),$$

is a Banach space. When $Re(H)^1$ is "complexified" then it is named **Hardy space** and denoted by $H^1(\mathbb{R})$, that is to say

$$f \in H^1(\mathbb{R}) \text{ if and only if } f = g + ih, \;\; g, h \in Re(H)^1.$$

The sharp maximal function of a locally integrable function f over $\mathbb{R}$ is defined by

$$f^{\#}(x) \equiv \sup \frac{1}{I} \int_I |f(y) - f_I|\, dy,$$

where the supremum is taking over all intervals $I = (a, b)$ such that $x \in I$ and

$$f_I \equiv \frac{1}{|I|} \int_I f(y) dy.$$

Let $\mathbf{B}$ be the set of all locally integrable functions f such that $f^{\#} \in L^\infty(\mathbb{R})$, and

$$\|f\|_* = \left\|f^{\#}\right\|_\infty .$$

So we obtain a seminorm $\|\cdot\|_*$ such that $\|f\|_* = 0$ if and only if f is a constant function almost everywhere.

In [9] we can find the following definition.

Definition 2. *The quotient space of* $\mathbf{B}$ *modulus the constant functions is called* ***Bounded Mean Oscillation*** *space functions and is denoted by* $BMO \equiv BMO(\mathbb{R})$ *and* $\|\cdot\|_*$ *is a norm over this space, making* $(BMO, \|\cdot\|_*)$ *a Banach space. Also,* $f \in BMO$ *when it determines one of the equivalence classes in this quotient space.*

This function space was introduced by John and Niremberg in [9], contains all bounded functions, that is to say, $L^\infty \subset BMO$, and the unbounded functions for which holds

$$|\{x \in I : |f(x) - f_I| > t\}| < C_1 e^{\left(\frac{-C_2 t}{\|f_*\| |I|}\right)},$$

for some $C_1, C_2 > 0$. This last inequality is known as ***John-Niremberg inequality***.

C. Fefferman in [1] established that **BMO** is the dual space of $H^1(\mathbb{R})$, also A. Torchinsky enunciated and proved, in [12], this same result.

Theorem 6. *$BMO(\mathbb{R}^n)$ is the dual space of the Hardy space $H^1(\mathbb{R}^n)$. The inner product $\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)g(x)dx$ for $f \in BMO$ and g belonging to the dense subspace of C^∞ rapidly decreasing functions in H^1.*

Other function spaces that will be of interest in this work are defined below. (See [10]).

Definition 3. *For $0 < p < \infty,\ 0 < q \leq \infty$ and $s \in \mathbb{R}$, the Homogeneus Triebel Lizorkin $\dot{F}_p^{s,q}$ space function is defined as the space consisting of all distributions (modulus polinomia) such that*

$$\left(\int_{\mathbb{R}}\left(\sum_{j\in\mathbb{Z}} 2^{js}\left|\varphi_{2^{-j}} * f(x)\right|^q\right)^{p/q} dx\right)^{1/p} < \infty,$$

where φ is a function that satisfy the following conditions

(i) $\varphi \in S$,

(ii) *supp* $(\hat{\varphi}) \subseteq \left\{\xi \in \mathbb{R} : \frac{1}{2} \leq |\xi| \leq 2\right\}$,

(iii) $\sum_{j\in\mathbb{Z}} \left|\hat{\varphi}\left(2^{-j}x\right)\right|^2 = 1$.

For $p = \infty,\ s \in \mathbb{R}$ and $0 < q \leq \infty$ the space $\dot{F}_\infty^{sq}$ is the set consisting of all tempered distributions (modulus polinomial) such that

$$\sup_{l,m\in\mathbb{Z}}\left(2^l \int_{\mathbb{R}} 2^{js} \sum_{j\in\mathbb{Z}} \left|\varphi_{2^{-j}} * f(x)\right|^q dx\right)^{1/q} < \infty,$$

where φ is a function that satisfy the conditions (i), (ii) and (iii).

In the following, $\Re_\infty^{0,2}$ will denote the set of all functions that satisfy (i), (ii), (iii) conditions in Definition 3.

For the purpose of this work is important next theorem, which is enunciated and proved by Jawerth and Frazier in [3].

Theorem 7. *Suppose that* $s \in \mathbb{R}$ *and* $0 < q < \infty$. *Then* $\left(\dot{F}_1^{s,q}\right)^* \approx \dot{F}_\infty^{-s,q'}$. *That is, if* $g \in \dot{F}_\infty^{-s,q'}$, *the mapping* l_g, *defined by* $l_g(f) = \langle f, g\rangle$, *for* $f \in S_0$, *where*

$$S_0 = \left\{ f : \widehat{f} = 0 \text{ in some open set containing zero} \right\},$$

can be extended to bounded linear functional over $\dot{F}_1^{s,q}$ *with* $\|l_g\| \approx \|g\|_{\dot{F}_\infty^{-s,q'}}$. *Reciprocally, if* $l \in \left(\dot{F}_1^{s,q}\right)^*$ *then there exist* $g \in \dot{F}_\infty^{-s,q'}$ *such that* $l = l_g$.

Also, Frazier et. al. in [5] proved the following result.

Theorem 8. *The Hardy space* H^1 *is isomorphic to the Triebel Lizorkin space* $\dot{F}_1^{0,2}$.

Remark 2. *The* BMO *space is isomorphic to the Triebel Lizorkin space* $\dot{F}_\infty^{0,2}$ *due to Theorems 6, 7 and 8.*

3 Main results

The following result is about Hardy-Littlewood maximal function.

Theorem 9. *Let* $\{f_n\}_{n\in\mathbb{Z}}$ *be a sequence of locally integrable functions. Then there exist* $C > 0$ *such that*

$$\sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{i\geq l} |\mathcal{M}f_i(x)|^2 \, dx \right)^{1/2} \leq C \sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{i\geq l} |f_i(x)|^2 \, dx \right)^{1/2},$$

where $I_{l,m} = \left[2^{-l}m,\ 2^{-l}(m+1)\right]$ *for* $l, m \in \mathbb{Z}$.

Proof. Let $f_i : i = 1, 2, 3, \ldots;$ be a sequence of locally integrable functions. From inequality (1) it is obtained that

$$\|\mathcal{M}(f_i)\|_p \leq C_p \|f_i\|_p, \ \ p > 1, \ i = 1, 2, \ldots$$

In particular if $p = 2$ then

$$\int_{\mathbb{R}} \mathcal{M}(f_i(x))^2 dx \leq C_2^2 \int_{\mathbb{R}} |f_i(x)|^2 \, dx.$$

Let $I \subset \mathbb{R}$ be an interval. Let $g_i : i = 1, 2, 3, \ldots$; be the sequence of locally integrable functions defined by $g_i = f_i \cdot \chi_I, \; i = 1, 2, 3, \ldots$ From (1) it is had

$$\int_{\mathbb{R}} (\mathcal{M} g_i(x))^2 dx \leq C_2^2 \int_{\mathbb{R}} |g_i(x)|^2 \, dx,$$

that is

$$\int_{\mathbb{R}} (\mathcal{M}(f_i \cdot \chi_I)(x))^2 dx \leq C_2^2 \int_{\mathbb{R}} |f_i \cdot \chi_I(x)|^2 \, dx = C_2^2 \int_I |f_i(x)|^2 \, dx,$$

therefore

$$\int_I (\mathcal{M}(f_i)(x))^2 dx \leq C_2^2 \int_I |f_i(x)|^2 \, dx.$$

Let $l, m \in \mathbb{Z}$ arbitrary and fixed integers. We can observe that if $I = I_{l,m}$ then

$$\begin{aligned} \int_{I_{l,m}} \sum_{i \geq l} (\mathcal{M} f_i(x))^2 dx &= \sum_{i \geq l} \int_{I_{l,m}} (\mathcal{M} f_i(x))^2 dx \qquad (2) \\ &\leq C_2^2 \sum_{i \geq l} \int_{I_{l,m}} |f_i(x)|^2 \, dx = C_2^2 \int_{I_{l,m}} \sum_{i \geq l} |f_i(x)|^2 \, dx. \end{aligned}$$

So, multiplying both sides of (2) the inequality by 2^l and taking supreme over $l, m \in \mathbb{Z}$ it is obtained

$$\sup_{l,m \in \mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{i \geq l} |\mathcal{M} f_i(x)|^2 \, dx \right)^{1/2} \leq C \sup_{l,m \in \mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{i \geq l} |f_i(x)|^2 \, dx \right)^{1/2},$$

where $C = C_2$. The proof is complete. □

Theorem 10. *Let $\psi \in \Re_\infty^{0,2}$. Then*

$$\sup_{l,m \in \mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{I_{j,k} \subset l_{l,m}} 2^j \left| \langle f, \psi_{j,k} \rangle \right|^2 \chi_{I_{j,k}}(x) \right) \leq C \, \|f\|_{\dot{F}_\infty^{0,2}},$$

for all distribution $f \in \dot{F}_\infty^{0,2}$.

Before proceed with the proof, let us stablish the following observation about the notation; the symbol

$$\sum_{I_{j,k} \subset l_{l,m}},$$

can be written as

$$\sum_{j \geq l} \sum_{k=2^{j-l}m}^{2^{j-l}(m+1)-1},$$

since $I_{j,k}$ and $I_{l,m}$ are dyadic intervals, and $I_{j,k} \subseteq I_{l,m}$ indicates that the sum is taken over those index $j, k \in \mathbb{Z}$ such that the inclusion condition hold.

Proof. Let us fix $l, m \in \mathbb{Z}$. Let $j \in \mathbb{Z}$ such that $j \geq l$ and $x \in I_{l,m}$. Using Proposition 1 we find

$$\begin{aligned}
&\sum_{k=2^{j-l}m}^{2^{j-l}(m+1)-1} \left|\langle f, \psi_{j,k}\rangle\right|^2 2^j \, \chi_{I_{j,k}}(x) \\
\leq &\sum_{k=2^{j-l}m}^{2^{j-l}(m+1)-1} \left(\sup_{y \in I_{j,k}} \left| \left(\widetilde{\psi}_{2^{-j}} * f \right)(y) \right| \right)^2 \chi_{I_{j,k}}(x).
\end{aligned} \tag{3}$$

Note that if $x \in I_{l,m}$, then there exists an unique $k' \in \left\{2^{j-l}m, \ldots, 2^{j-l}(m+1)-1\right\}$ such that $x \in I_{j,k'}$, because of $I_{j,k}$ are disjoint intervals. Then the inequality (3) can be rewritten as

$$\sum_{k=2^{j-l}m}^{2^{j-l}(m+1)-1} \left|\langle f, \psi_{j,k}\rangle\right|^2 2^j \, \chi_{I_{j,k}}(x) \leq \left(\sup_{y \in I_{j,k}} \left| \left(\widetilde{\psi}_{2^{-j}} * f \right)(y) \right| \right)^2 \chi_{I_{j,k'}}(x). \tag{4}$$

But, if $x \in I_{j,k'}$ then for all $y \in I_{j,k'}$ there exists $z \in \mathbb{R}$ such that $y = x - z$, and since

$$2^{-j}k' \leq x \leq 2^{-j}(k'+1) \quad \text{and} \quad 2^{-j}k' \leq y \leq 2^{-j}(k'+1),$$

then

$$2^{-j}k' - 2^{-j}(k+1) \leq x - y \leq 2^{-j}(k'+1) - 2^{-j}k',$$

that is

$$-2^{-j} \leq z \leq 2^{-j},$$

which is equivalent to

$$|z| \leq 2^{-j},$$

so, the inequality (4) can be written as

$$\sum_{k=2^{j-l}m}^{2^{j-l}(m+1)-1} |\langle f, \psi_{j,k} \rangle|^2 \, 2^j \, \chi_{I_{j,k}}(x) \leq \left(\sup_{|z| \leq 2^{-j}} \left| \left(\widetilde{\psi}_{2^{-j}} * f \right)(x-z) \right| \right)^2 . \quad (5)$$

Let λ be an arbitrary positive real number. Multiplying and dividing by $\left(1 + 2^j \, |z|\right)^{2\lambda}$ in inequality (5), it is had

$$\begin{aligned}
\sum_{k=2^{j-l}m}^{2^{j-l}(m+1)-1} |\langle f, \psi_{j,k} \rangle|^2 \, 2^j \, \chi_{I_{j,k}}(x) &\leq \left(\frac{\sup\limits_{|z| \leq 2^{-j}} \left| \left(\widetilde{\psi}_{2^{-j}} * f \right)(x-z) \right|}{\left(1 + 2^j \, |z|\right)^{\lambda}} \right)^2 \left(1+2^j|z|\right)^{2\lambda} \\
&\leq \left(\frac{\sup\limits_{|z| \leq 2^{-j}} \left| \left(\widetilde{\psi}_{2^{-j}} * f \right)(x-z) \right|}{\left(1 + 2^j \, |z|\right)^{\lambda}} \right)^2 2^{2\lambda} \\
&= \left(2^{\lambda} \left(\psi_{j,\lambda}^{**} f \right)(x) \right)^2,
\end{aligned}$$

where $\psi_{j\lambda}^{**} f$ is the maximal function defined in Lemma 5.

Taking the sum respect to j it is obtained

$$\sum_{j \geq l} \sum_{k=2^{j-l}m}^{2^{j-l}(m+1)-1} |\langle f, \psi_{j,k} \rangle|^2 \, 2^j \, \chi_{I_{j,k}}(x) \leq \sum_{j \geq l} \left(2^{\lambda} \left(\psi_{j,\lambda}^{**} f \right)(x) \right)^2, \quad (6)$$

in consequence, using the Lemma 5

$$\sum_{j \geq l} \sum_{k=2^{j-l}m}^{2^{j-l}(m+1)-1} |\langle f, \psi_{j,k} \rangle|^2 \, 2^j \, \chi_{I_{j,k}}(x) \leq C_{\lambda} \sum_{j \geq l} \left\{ \mathcal{M} \left(|\psi_{2^{-j}} * f|^{1/\lambda} \right)(x) \right\}^{2\lambda}.$$

In particular, taking $\lambda = 1$ and using the Theorem 9 with $f_j = \psi_{2^{-j}} * f$, it follows

$$C_1 \int_{I_{l,m}} \sum_{j\geq l} \{\mathcal{M}\left(|\psi_{2^{-j}} * f|\right)(x)\}^2 \, dx \leq C_1 C \int_{I_{l,m}} \sum_{j\geq l} |\left(\psi_{2^{-j}} * f\right)(x)|^2 \, dx,$$

so

$$\int_{I_{l,m}} \sum_{j\geq l} \sum_{k=2^{j-l}m}^{2^{j-l}(m+1)-1} |\langle f, \psi_{j,k}\rangle|^2 \, 2^j \, \chi_{I_{j,k}}(x) \leq C_1 C \int_{I_{l,m}} \sum_{j\geq l} |\left(\psi_{2^{-j}} * f\right)(x)|^2 \, dx,$$

multiplying by 2^l, and taking supreme over $l, m \in \mathbb{Z}$, it is attained

$$\sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{j\geq l} \sum_{k=2^{j-l}m}^{2^{j-l}(m+1)-1} |\langle f, \psi_{j,k}\rangle|^2 \, 2^j \, \chi_{I_{j,k}}(x) \right) \leq C \, \|f\|_{\dot{F}^{0,2}_{\infty}} .$$

Here, the capital letter C is used for denote the product $C_1 C$. The proof is complete. $\square$

Next result establishes an opposite inequality to that found in Theorem 10.

Theorem 11. *Let ψ be a function in $\Re^{0,2}_{\infty}$. If $f \in \dot{F}^{0,2}_{\infty}$ then*

$$\|f\|_{\dot{F}^{0,2}_{\infty}} \leq D \sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{j\geq l} \sum_{k=2^{j-l}m}^{2^{j-l}(m+1)-1} |\langle f, \psi_{j,k}\rangle|^2 \, 2^j \, \chi_{I_{j,k}}(x) \right) .$$

Proof. Let $\psi \in \Re^{0,2}_{\infty}$ and $f \in \dot{F}^{0,2}_{\infty}$ be as in the statement. Since f is a tempered distribution (modulus polinomial), by Lemma 1, it can be written as

$$f = \sum_{r\in\mathbb{Z}} \sum_{s\in\mathbb{Z}} \langle f, \psi_{r,s}\rangle \, \psi_{r,s}.$$

For a fixed and arbitrary $j \in \mathbb{Z}$, it follows

$$\left(\psi_{2^{-j}} * f\right)(x) = \sum_{r\in\mathbb{Z}} \sum_{s\in\mathbb{Z}} \langle f, \psi_{r,s}\rangle \left(\psi_{2^{-j}} * \psi_{r,s}\right)(x),$$

and taking Fourier transform in that expression, it is obtained

$$\left(\psi_{2^{-j}} * f\right)^{\wedge}(\xi) = \sum_{r\in\mathbb{Z}} \sum_{s\in\mathbb{Z}} \langle f, \psi_{r,s}\rangle \, \hat{\psi}_{2^{-j}}(\xi) \hat{\psi}_{r,s}(\xi). \tag{7}$$

Now, it is known that supp $\left(\widehat{\psi}\right) \subseteq \left\{\xi \in \mathbb{R} : \left(\frac{1}{2}\right) \leq |\xi| \leq 2\right\}$ implies that

$$\operatorname{supp}\left(\widehat{\psi}_{2^{-j}}\right) \subseteq \left\{\xi \in \mathbb{R} : 2^{j-1} \leq |\xi| \leq 2^{j+1}\right\},$$

and

$$\operatorname{supp}\left(\widehat{\psi}_{r,s}\right) \subseteq \left\{\xi \in \mathbb{R} : 2^{r-1} \leq |\xi| \leq 2^{r+1}\right\},$$

therefore, if $\xi \notin \operatorname{supp}\left(\widehat{\psi}_{2^{-j}}\right) \bigcap \operatorname{supp}\left(\widehat{\psi}_{r,s}\right)$ then the terms $\hat{\psi}_{2^{-j}}(\xi)\hat{\psi}_{r,s}(\xi)$ are zeroes for those $r \in \mathbb{Z}$ such that $r \notin \{j-1, j, j+1\}$, so the series (7) can be written as

$$\left(\psi_{2^{-j}} * f\right)^{\wedge}(\xi) = \sum_{r=j-1}^{j+1} \sum_{s\in\mathbb{Z}} \langle f, \psi_{r,s}\rangle\, \hat{\psi}_{2^{-j}}(\xi)\hat{\psi}_{r,s}(\xi).$$

Applying Fourier inverse transform it is had

$$\left(\psi_{2^{-j}} * f\right)(x) = \sum_{r=j-1}^{j+1} \sum_{s\in\mathbb{Z}} \langle f, \psi_{r,s}\rangle \left(\psi_{2^{-j}} * \psi_{r,s}\right)(x).$$

Since $\psi \in S$ then $\left(\psi_{2^{-j}} * \psi_{r,s}\right) \in S$, and for $\varepsilon > 0$, an application of Lemma 2, it follows that

$$\left|\psi_{2^{-j}} * \psi_{r,s}(x)\right| \leq \frac{2^{\frac{j}{2}}C}{\left(1 + 2^r\left|x - 2^{-r}s\right|\right)^{1+\varepsilon}}.$$

Since $g(x) = \dfrac{1}{\left(1 + |x-a|\right)^{1+\varepsilon}}$ has an absolute maximum in $x = a$, then, if $x \in I_{j-1,s_1} \subseteq I_{j,s_2} \subseteq I_{j+1,s_3}$ it is obtained

$$\frac{2^{\frac{j}{2}}C}{\left(1 + 2^{j-1}\left|x - 2^{-j+1}s\right|\right)^{1+\varepsilon}} \leq \frac{2^{\frac{j}{2}}C}{\left(1 + 2^{j-1}\left|2^{j-1}s_1 - 2^{-j+1}s\right|\right)^{1+\varepsilon}},$$
$$\frac{2^{\frac{j}{2}}C}{\left(1 + 2^{j}\left|x - 2^{-j}s\right|\right)^{1+\varepsilon}} \leq \frac{2^{\frac{j}{2}}C}{\left(1 + 2^{j}\left|2^{j}s_2 - 2^{-j}s\right|\right)^{1+\varepsilon}},$$

and

$$\frac{2^{\frac{j}{2}}C}{\left(1 + 2^{j}\left|x - 2^{-j}s\right|\right)^{1+\varepsilon}} \leq \frac{2^{\frac{j}{2}}C}{\left(1 + 2^{j+1}\left|2^{j+1}s_3 - 2^{-j-1}s\right|\right)^{1+\varepsilon}},$$

for all $s \in \mathbb{Z}$.

In consequence,

$$
\begin{aligned}
|\psi_{2^{-j}} * f(x)|\,|\psi_{2^{-j}} * f(x)| \leq & \sum_{s\in\mathbb{Z}} \frac{2^{\frac{j}{2}} C\,|\langle f, \psi_{j-1,s}\rangle|}{(1+2^{j-1}\,|2^{j-1}s_1 - 2^{-j+1}s|)^{1+\varepsilon}} \\
& + \sum_{s\in\mathbb{Z}} \frac{2^{\frac{j}{2}} C\,|\langle f, \psi_{j,s}\rangle|}{(1+2^{j}\,|2^{j}s_2 - 2^{-j}s|)^{1+\varepsilon}} \\
& + \sum_{s\in\mathbb{Z}} \frac{2^{\frac{j}{2}} C\,|\langle f, \psi_{j+1,s}\rangle|}{(1+2^{j+1}\,|2^{j+1}s_3 - 2^{-j-1}s|)^{1+\varepsilon}}.
\end{aligned}
$$

Now, with an application of Lemma 3 to each sum, it follows estimate

$$
\begin{aligned}
|\psi_{2^{-j}} * f(x)| \leq 2^{\frac{j}{2}} C \Bigg[& \mathcal{M}\left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j-1,s}\rangle|\,\chi_{I_{j-1,s}}\right)(x) \\
& + \mathcal{M}\left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j,s}\rangle|\,\chi_{I_{j,s}}\right)(x) \\
& + \mathcal{M}\left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j+1,s}\rangle|\,\chi_{I_{j+1,s}}\right)(x) \Bigg].
\end{aligned}
$$

Taking power operations, summing about j, integrating over $I_{l,m}$ and taking supreme over $l, m \in \mathbb{Z}$ it is attained

$$
\begin{aligned}
\|f\|_{\dot{F}^{0,2}_{\infty}} &= \sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{j\geq l} |\psi_{2^{-j}} * f(x)|^2\,dx\right)^{1/2} \\
&\leq C \left[\sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{j\geq l} \mathcal{M}^2\left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j-1,s}\rangle|\,\chi_{I_{j-1,s}}\right)(x)dx\right)^{1/2} \right. \\
&\quad + \sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{j\geq l} \mathcal{M}^2\left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j,s}\rangle|\,\chi_{I_{j,s}}\right)(x)dx\right)^{1/2} \\
&\quad \left. + \sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{j\geq l} \mathcal{M}^2\left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j+1,s}\rangle|\,\chi_{I_{j+1,s}}\right)(x)dx\right)^{1/2} \right]. \qquad (8)
\end{aligned}
$$

Now, doing the index change $k = j - 1$ in the first term of the previous sum,

$$2^l \int_{I_{l,m}} \sum_{j\geq l} \mathcal{M}^2 \left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j-1,s}\rangle|\, 2^{\frac{j}{2}} \chi_{I_{j-1,s}} \right)(x)dx$$
$$= 2^l \int_{I_{l,m}} \sum_{k\geq l-1} \mathcal{M}^2 \left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{k,s}\rangle|\, 2^{\frac{j}{2}} \chi_{I_{k,s}} \right)(x)dx,$$

since $I_{l,m} \subset I_{l-1,m-1}$, it is had

$$2^l \int_{I_{l,m}} \sum_{j\geq l} \mathcal{M}^2 \left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j-1,s}\rangle|\, 2^{\frac{j}{2}} \chi_{I_{j-1,s}} \right)(x)dx$$
$$\leq 2^l C \int_{I_{l-1,m-1}} \sum_{j\geq l} \mathcal{M}^2 \left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{k,s}\rangle|\, 2^{\frac{k}{2}} \chi_{I_{k,s}} \right)(x)dx,$$

so, renaming the index in the series and taking supreme over $l, m \in \mathbb{Z}$

$$\sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{j\geq l} \mathcal{M}^2 \left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j+1,s}\rangle|\, 2^{\frac{j}{2}} \chi_{I_{j+1,s}} \right)(x)dx \right)^{1/2}$$
$$\leq C \sup_{l,m\in\mathbb{Z}} \left(2^{l-1} \int_{I_{l-1,m-1}} \sum_{j\geq l} \mathcal{M}^2 \left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j,s}\rangle|\, 2^{\frac{j}{2}} \chi_{I_{j,s}} \right)(x)dx \right)^{1/2}.$$

Now, adding the positive terms of the form $|\langle f, \psi_{j,s}\rangle|$ with $s \in \mathbb{Z}$ in the third term of (8), it is seen that

$$\sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{j\geq l} \mathcal{M}^2 \left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j-1,s}\rangle|\, 2^{\frac{j}{2}} \chi_{I_{j-1,s}} \right)(x)dx \right)^{1/2}$$
$$\leq C \sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l-1,m-1}} \sum_{j\geq l} \mathcal{M}^2 \left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j,s}\rangle|\, 2^{\frac{j}{2}} \chi_{I_{j,s}} \right)(x)dx \right)^{1/2}.$$

Thus inequality (8) remains as

$$\|f\|_{\dot{F}^{0,2}_{\infty}} \leq C \sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l-1,m-1}} \sum_{j\geq l} \mathcal{M}^2 \left(\sum_{s\in\mathbb{Z}} |\langle f, \psi_{j,s}\rangle|\, 2^{\frac{j}{2}} \chi_{I_{j,s}} \right)(x)dx \right)^{1/2}.$$

Applying Theorem 9 in the right side of the previous inequality, it is attained

$$\|f\|_{\dot{F}^{0,2}_{\infty}} \leq C \sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l-1,m-1}} \sum_{j\geq l} \sum_{s\in\mathbb{Z}} |\langle f, \psi_{j,s}\rangle|^2 \, 2^{\frac{j}{2}} \chi_{I_{j,s}}(x) dx \right)^{1/2}.$$

The proof is complete. □

The following proposition shows another valid expression for the considered characterization.

Proposition 2.

$$\sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f, \psi_{j,s}\rangle|^2 \, 2^{\frac{j}{2}} \chi_{I_{j,s}}(x) dx \right)^{1/2} < \infty,$$

if and only if there exist a $C > 0$ such that

$$2^l \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f, \psi_{j,s}\rangle|^2 \leq C,$$

for all $l, m \in \mathbb{Z}$.

Proof. If

$$\sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f, \psi_{j,s}\rangle|^2 \, 2^{\frac{j}{2}} \chi_{I_{j,s}}(x) dx \right)^{1/2} = A < \infty,$$

then

$$2^l \int_{I_{l,m}} \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f, \psi_{j,s}\rangle|^2 \, 2^{\frac{j}{2}} \chi_{I_{j,s}}(x) \, dx \ \leq A^2,$$

for all $m, l \in \mathbb{Z}$. Since

$$\int_{I_{l,m}} \chi_{I_{j,s}}(x) dx = 2^{\cdot j},$$

it is obtained that

$$\begin{aligned} &2^l \int_{I_{l,m}} \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f, \psi_{j,s}\rangle|^2 \, 2^{\frac{j}{2}} \chi_{I_{j,s}}(x) dx \\ =&2^l \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f, \psi_{j,s}\rangle|^2 \, 2^{\frac{j}{2}} \int_{I_{l,m}} \chi_{I_{j,s}}(x) dx \leq A^2. \end{aligned}$$

That is

$$2^l \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f,\psi_{j,s}\rangle|^2 \leq C,$$

where $C = A^2$.

Reciprocally, if there exist $C > 0$ such that

$$2^l \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f,\psi_{j,s}\rangle|^2 \leq C,$$

for all $l, m \in \mathbb{Z}$ then

$$\begin{aligned} 2^l \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f,\psi_{j,s}\rangle|^2\, 2^j 2^{-j} &= 2^l \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f,\psi_{j,s}\rangle|^2\, 2^{\frac{j}{2}} \int_{I_{l,m}} \chi_{I_{j,s}}(x)dx \\ &= 2^l \int_{I_{l,m}} \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f,\psi_{j,s}\rangle|^2\, 2^{\frac{j}{2}} \chi_{I_{j,s}}(x)dx, \end{aligned}$$

so

$$2^l \int_{I_{l,m}} \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f,\psi_{j,s}\rangle|^2\, 2^{\frac{j}{2}} \chi_{I_{j,s}}(x)dx \leq C.$$

Operating with the $\frac{1}{2}$ power and taking supreme over $l, m \in \mathbb{Z}$, it is had

$$\sup_{l,m\in\mathbb{Z}} \left(2^l \int_{I_{l,m}} \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f,\psi_{j,s}\rangle|^2\, 2^{\frac{j}{2}} \chi_{I_{j,s}}(x)dx \right)^{1/2} \leq C.$$

So, the proof is complete. □

As immediate consequence from Proposition 2, Theorems 10 and 11 we have the following.

Corollary 1. *Let $\psi \in \Re_\infty^{0,2}$. $f \in BMO$ if and only if there exist $C > 0$ such that*

$$2^l \sum_{I_{j,k}\subseteq I_{l,m}} |\langle f,\psi_{j,s}\rangle|^2 \leq C,$$

for all $l, m \in \mathbb{Z}$.

In particular, if ψ is a orthonormal wavelet in $\Re_\infty^{0,2}$ we obtain a characterization of BMO using wavelets coefficients. The *Lemorié - Meyer* wavelet, showed Preliminaries section, is in the class $\Re_\infty^{0,2}$.

4 Conclusion

In the present study, results concerning the characterization of the space BMO functions are found by using a special class of wavelet and homogeneous spaces of Triebel Lizorkin $\dot{F}_{\infty}^{0,2}$. Also, a new version of a vector-valued inequality using the classic maximal function of Hardy-Littlewood was also found (Theorem 9). Finally, as a consequence of the characterization established for BMO (Theorems 10 and 11), a result was found (Corollary 1) which responses to a proposal made by Wojtaszczyk in [14].

This work is expected to serve as a useful and motivating tool to find new characterizations of function spaces using wavelets.

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