© Revista de Matemática: Teoría y Aplicaciones 2021 **28**(1) : 79–94
cimpa – ucr issn: 1409-2433 (Print), 2215-3373 (Online)
doi: https://doi.org/10.15517/rmta.v28i1.36294

HYDROSTATIC LIMIT FOR THE SYMMETRIC EXCLUSION PROCESS WITH LONG JUMPS: SUPPER-DIFFUSIVE CASE

LÍMITE HIDROESTÁTICO PARA EL PROCESO DE EXCLUSIÓN CON SALTOS LARGOS: CASO SUPER-DIFUSIVO

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Received: 28/Jun/2019; Revised: 04/Nov/20;
Accepted: 30/Oct/2020

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Abstract

Hydrostatic behavior for the one dimensional exclusion process with long jumps in contact with infinite reservoirs at different densities are derived. The jump rate is described by a transition probability p which is proportional to $|\cdot|^{-(\gamma+1)}$ for $1 < \gamma < 2$ (supper-diffusive case). The reservoirs add or remove particles with rate proportional to $\kappa > 0$.

Keywords: exclusion process with long jumps; super-diffusion; fractional Fick's law.

Resumen

Se deriva el comportamiento hidroestático del proceso de exclusión simple con saltos largos en contacto con depósitos infinitos con diferentes densidades. La tasa de salto es descrita por una función de probabilidad p que es proporcional a $|\cdot|^{-(\gamma+1)}$ para $1 < \gamma < 2$ (caso súper-difusivo). Los depósitos de partículas añaden o retiran partículas con una tasa propocional a $\kappa > 0$.

Palabras clave: proceso de exclusión con salto largos; super-difusión; ley de Fick fraccionaria.

Mathematics Subject Classification: 82C22, 60K35, 34A08.

1 Introduction

In the microscopic world the particles behave chaotically. By contrast, the behavior and properties in the macroscopic world are deterministic. The most interesting is that microscopic behavior are reflected in macroscopic properties. At this point is where thermodynamic and statistical mechanics appears in order to explain the relation between these two worlds. Thermodynamic helps to describe the properties of macroscopic system near to the equilibrium (there is no net macroscopic flow of matter or energy within the system). For that purpose is necessary a small number of macroscopic variables (thermodynamic characteristics) such as temperature, density, pressure and others. Statistical mechanics aims to understand the thermodynamic characteristics of a system on the basis of its microscopical behavior. For that purpose statistical mechanics uses the probability theory. In order to understand the role of the probability in statistical mechanics we flip N times a fair coin. There are 2^N possible head and tail configurations, it means that, the behavior of the system is very chaotic and unruly but thanks to the probability we know that for a large amount of coin tosses the behavior of the system is predictable.

In this work we consider the symmetric exclusion process with long jumps on $\Lambda_N := \{1, \cdots, N-1\}$ in contact with infinitely many stochastic reservoirs. Each pair of sites of the bulk $\{x, y\} \subset \Lambda_N$ carries a Poisson process of intensity one. The Poisson processes associated to different bonds are independent. When the clock associated to $\{x, y\}$ rings, the particles at the sites are exchanged with rate $p(y-x)$, if one of the site is empty and the other one is not. Otherwise nothing happens. In the dynamics at the left boundary each pair of sites $\{x, y\}$ with $x \in \Lambda_N$ and $y \in \mathbb{Z}_-$ carries a Poisson process of intensity one, where they are all independent. If the clock associated to the bound $\{x, y\}$ rings, then a particle can get into (resp. get out from) the bulk from (resp. to) the left reservoir at rate $\alpha\kappa p(z)$ (resp. $(1-\alpha)\kappa p(z)$) where $z = y - x$ is the size of the jump, if the site at x is empty (resp. occupied). The right reservoir acts in the same way, except that α is replaced by β in the jump rates given above. In this paper we are interested in the case where $p(\cdot)$ has a long tail, proportional to $|\cdot|^{-(1-\gamma)}$ for $\gamma \in (1, 2)$.

The aim of this work is to generalize the stationary scenario given in [3] with a new family operator indexed by κ, defined by its action on smooth functions G with compact support on $(0, 1)$ by

$$\mathbb{L}_\kappa G(u) := c_\gamma \lim_{\varepsilon \to 0} \int_0^1 \mathbf{1}_{|u-v| \geq \varepsilon} \frac{G(v) - G(u)}{|u-v|^{1+\gamma}} dv - \kappa\gamma^{-1}(u^{-\gamma} + (1-u)^{-\gamma}),$$

for $u \in (0, 1)$ and where c_γ is defined in section 2. For these kind of function G these operator are symmetric non-positive. In the case $\kappa = 1$ we obtain the restricted fractional Laplacian. In [3] is studied the hydrostatic limit and the fractional Fick's law. In this paper we explain that these results can be generalized for the operator $\mathbb{L}_\kappa$. Even though we deal with the same results using similar methods (just for the Fick's law), we are very interested in divulging the results and these kind of operators to the community.

The outline of this paper is as follows. In Section 2 we describe the model more precisely. We introduce the hydrodynamic equations and state the results. In Section 3 we deal with hydrostatic limit and Fick's law. Finally, we present the proof for the hydrostatic limit.

2 Notation and results

2.1 The model

For an integer $N \geq 2$ let $\Lambda_N = \{1, \ldots, N-1\}$ and $\Omega_N = \{0, 1\}^{\Lambda_N}$. Fix $\gamma \in (1, 2)$. Let $p(\cdot)$ be a probability on $\mathbb{Z}$ defined by

$$p(z) = c_\gamma \frac{\mathbf{1}_{z\neq 0}}{|z|^{\gamma+1}}, \tag{1}$$

where $c_\gamma^{-1} = 2\zeta_{\gamma+1}$ (ζ_s is the Riemann zeta function defined for $s > 1$).

Fix $0 \leq \alpha \leq \beta \leq 1$ and $\kappa > 0$. We consider the symmetric long jumps exclusion process on Λ_N with infinitely many stochastic reservoirs with density α at all negative integer sites $j \leq 0$ and with density β at all integer sites $j \geq N$ (see [1] for more details about the model).

The process is characterized by its infinitesimal generator

$$L_N = L_N^b + \kappa \left[L_N^\ell + L_N^r \right],$$

where the generator L_N^b corresponds to the bulk dynamics and generators L_N^ℓ and L_N^r corresponding to non-conservative boundary dynamics. The action of L_N on functions $f : \Omega_N \to \mathbb{R}$ is given by

$$\begin{aligned}
(L_N^b f)(\eta) &= \frac{1}{2} \sum_{x,y\in\Lambda_N} p(x-y)[f(\eta^{xy}) - f(\eta)], \\
(L_N^\ell f)(\eta) &= \sum_{\substack{x\in\Lambda_N \\ y\leq 0}} p(x-y)c_x(\eta;\alpha)[f(\eta^x) - f(\eta)], \\
(L_N^r f)(\eta) &= \sum_{\substack{x\in\Lambda_N \\ y\geq N}} p(x-y)c_x(\eta;\beta)[f(\eta^x) - f(\eta)],
\end{aligned} \tag{2}$$

where

$$(\eta^{xy})_z = \begin{cases} \eta_z, \ z \neq x, y, \\ \eta_y, \ z = x, \\ \eta_x, \ z = y. \end{cases} \qquad (\eta^x)_z = \begin{cases} \eta_z, \ z \neq x, \\ 1 - \eta_x, \ z = x, \end{cases}$$

and for any $x \in \Lambda_N$ and any $\eta \in \Omega_N$ we have that

$$c_x(\eta;\alpha) = [\eta_x(1-\alpha) + (1-\eta_x)\alpha]$$

and

$$c_x(\eta;\beta) = [\eta_x(1-\beta) + (1-\eta_x)\beta].$$

Given $x \in \Lambda_N \cup \{N\}$ and a configuration η, we denote by $W_x(\eta)$ to the current over the value $x - \frac{1}{2}$, which is defined as the rate of particles crossing

$x-\frac{1}{2}$ from the left to the right, minus the rate of particles crossing $x-\frac{1}{2}$ from the right to the left. Then, the current can be written as

$$\begin{aligned} W_x(\eta) = & \sum_{\substack{1\leq y\leq x-1\\ x-1<z\leq N-1}} p(z-y)(\eta_y-\eta_z) \\ & +\kappa\Bigg[\sum_{\substack{x\leq z\leq N-1\\ y\leq 0}} p(z-y)(\alpha-\eta_z) - \sum_{\substack{1\leq y\leq x-1\\ z\geq N}} p(z-y)(\beta-\eta_y)\Bigg] \\ & =:W_x^b(\eta)+\kappa W_x^{\ell,r}(\eta). \end{aligned} \tag{3}$$

We will often omit the dependence of W_x on η. We can observe that for any $x\in\Lambda_N$ we have that $L_N\eta_x$ is equal to

$$\sum_{y\in\Lambda_N} p(y-x)[\eta_y-\eta_x]+\kappa\Big[\sum_{y\leq 0} p(y-x)(\alpha-\eta_x)+\sum_{y\geq N} p(y-x)(\beta-\eta_x)\Big].$$

Now, we also note that W_x-W_{x+1} is equal to

$$\begin{aligned} & \sum_{\substack{1\leq y\leq x-1\\ x-1<z\leq N-1}} p(z-y)(\eta_y-\eta_z) - \sum_{\substack{1\leq y\leq x\\ x+1<z\leq N-1}} p(z-y)(\eta_y-\eta_z) \\ & +\kappa\Bigg[\sum_{\substack{x\leq z\leq N-1\\ y\leq 0}} p(z-y)(\alpha-\eta_z) - \sum_{\substack{x+1\leq z\leq N-1\\ y\leq 0}} p(z-y)(\alpha-\eta_z)\Bigg] \\ & +\kappa\Bigg[\sum_{\substack{1\leq y\leq x\\ z\geq N}} p(z-y)(\beta-\eta_y) - \sum_{\substack{1\leq y\leq x-1\\ z\geq N}} p(z-y)(\beta-\eta_y)\Bigg]. \end{aligned}$$

Thus, it is not difficult to see that the microscopic continuity equation is

$$L_N\eta_x=-\nabla W_x:=-(W_{x+1}-W_x).$$

Let us denote by $\{\eta(t)\}_{t\geq 0}$ the Markov process associated to the generator L_N speeded up by N^γ, it means that, the process with generator $N^\gamma L_N$. For $\rho\in(0,1)$, we denote by ν_ρ the Bernoulli product measure in Ω_N with density ρ, that is, the measure whose marginals satisfy $\nu_\rho(\eta_x=1)=1-\nu_\rho(\eta_x=0)=\rho$. The irreducible Markov process generated by L_N has a unique invariant measure that we denote by $\bar\mu_N$ and $f_{N,\rho}$ denote its density with respect to the measure ν_ρ. If $\alpha=\beta=\rho$ then $\bar\mu_N=\nu_\rho$. To simplify the notation of the expectation with respect to $\bar\mu_N$ (resp. ν_ρ) we will often use the notation $\int_{\Omega_N} f(\eta)d\bar\mu_N(\eta)=\langle f\rangle_N$ $\Big($resp. $\int_{\Omega_N} f(\eta)d\nu_\rho(\eta)=\langle f\rangle_\rho\Big)$.

2.2 Notation

To properly state the hydrostatic limits, we need to introduce some notations and definitions. Firstly we abbreviate the Hilbert space $L^2([0,1]^d, h(u)du)$ for $d=1,2$, by $L^2_h([0,1]^d)$. Also, we denote its inner product by $\langle \cdot,\cdot\rangle_h$ and the corresponding norm by $\|\cdot\|_h$. When $h\equiv 1$ we simply write $L^2([0,1]^d)$, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. The set $C^\infty([0,1]^d)$ denotes the set of restrictions of smooth functions on $\mathbb{R}$ to $[0,1]^d$. We denote by $C_c^\infty((0,1)^d)$ the set of all smooth real-valued functions defined in $(0,1)^d$ with compact support contained in $(0,1)^d$.

Now, we recall that the fractional Laplacian $-(-\Delta)^{\gamma/2} := -(-\Delta)^{\gamma/2}_{\mathbb{R}}$ of exponent $\gamma/2$ is defined on the set of functions $G:\mathbb{R}\to\mathbb{R}$ such that

$$\int_{-\infty}^{\infty}\frac{|G(u)|}{(1+|u|)^{1+\gamma}}du<\infty, \tag{4}$$

by

$$-(-\Delta)^{\gamma/2}G\,(u)=c_\gamma\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\mathbf{1}_{|u-v|\geq\varepsilon}\frac{G(v)-G(u)}{|u-v|^{1+\gamma}}dv,$$

provided that the limit exists (which is the case, for example, if G is in the Schwartz space). The number c_γ is defined in (1). The operator $-(-\Delta)^{\gamma/2}$ is the generator of a γ-Lévy stable process, up to a multiplicative constant.

We define the operator $\mathbb{L}$ by its action on functions $G\in C^\infty((0,1))$, by

$$\forall u\in(0,1),\quad (\mathbb{L}G)(u)=c_\gamma\lim_{\varepsilon\to 0}\int_0^1\mathbf{1}_{|u-v|\geq\varepsilon}\frac{G(v)-G(u)}{|u-v|^{1+\gamma}}dv. \tag{5}$$

We can see that the right hand side of (5) is well defined by performing a second order Taylor expansion of G at u. We observe by a symmetry argument that, for ε sufficiently small we have that

$$\int_0^1\mathbf{1}_{|v-u|\geq\varepsilon}\frac{v-u}{|v-u|^{1+\gamma}}dv=\int_u^{1-u}\frac{v}{|v|^{1+\gamma}}dv,$$

and we conclude that remainder term is integrable. The operator $\mathbb{L}$ is called the *regional fractional Laplacian* on $(0,1)$. The semi inner-product $\langle\cdot,\cdot\rangle_{\gamma/2}$ is defined on the set $C^\infty((0,1))$ by

$$\langle G,H\rangle_{\gamma/2}=\frac{c_\gamma}{2}\iint_{[0,1]^2}\frac{(H(u)-H(v))(G(u)-G(v))}{|u-v|^{1+\gamma}}du\,dv.$$

The corresponding semi-norm is denoted by $\|\cdot\|_{\gamma/2}$. Observe that for any $G,H\in C^\infty((0,1))$ we have that

$$\langle G,-\mathbb{L}H\rangle=\langle -\mathbb{L}G,H\rangle=\langle G,H\rangle_{\gamma/2}.$$

In order to simplify the notation we define the functions $r_N^{\pm} : [0,1] \to \mathbb{R}$ such that for $x \in \Lambda_N$ as follows: at the points $\frac{x}{N}$ the are defined as

$$r_N^-(\tfrac{x}{N}) = \sum_{y \geq x} p(y), \quad r_N^+(\tfrac{x}{N}) = \sum_{y \leq x-N} p(y), \tag{6}$$

with $r_N^{\pm}(0) = r_N^{\pm}(\frac{1}{N})$ and $r_N^{\pm}(1) = r_N^{\pm}(\frac{N-1}{N})$. At the remaining points, we define it by linear interpolation. Let us consider the functions $r^{\pm} : (0,1) \to \mathbb{R}_+$ defined by $r^-(u) = c_\gamma \gamma^{-1} u^{-\gamma}$ and $r^+(u) = c_\gamma \gamma^{-1}(1-u)^{-\gamma}$. By Lemma 3.3 of [3] we know that

$$\lim_{N \to \infty} N^\gamma r_N^{\pm}(u) = r^{\pm}(u), \tag{7}$$

uniformly in any compact set contained in $(0,1)$. Moreover, we introduce the functions

$$\begin{aligned} V_1(u) &= r^-(u) + r^+(u), \\ V_0(u) &= \alpha r^-(u) + \beta r^+(u), \\ \rho^\infty(u) &= \frac{V_0(u)}{V_1(u)}. \end{aligned} \tag{8}$$

We also introduce a family of operators indexed by κ and defined by

$$\mathbb{L}_\kappa = \mathbb{L} - \kappa V_1. \tag{9}$$

We know that these operators are symmetric and non-positive when acts over $C_c^\infty((0,1))$. For $\kappa = 1$, we recover the so-called restricted fractional Laplacian (see [7]):

$$\forall u \in (0,1), \quad -(-\Delta)^{\gamma/2} G\,(u) = (\mathbb{L}G)(u) - V_1(u)G(u) := (\mathbb{L}_1 G)(u). \tag{10}$$

In the other hand, if we consider the limit $\kappa \to 0$ we get the regional fractional Laplacian.

Definition 1 *The Sobolev space $\mathcal{H}^{\gamma/2} := \mathcal{H}^{\gamma/2}([0,1])$ consists of all square integrable functions $g : (0,1) \to \mathbb{R}$ such that $\|g\|_{\gamma/2} < \infty$. This is a Hilbert space for the norm $\|\cdot\|_{\mathcal{H}^{\gamma/2}}$ defined by*

$$\|g\|^2_{\mathcal{H}^{\gamma/2}} := \|g\|^2 + \|g\|^2_{\gamma/2}.$$

The elements of this space coincide a.e. with continuous functions. The completion of $C_c^\infty((0,1))$ for this norm is denoted by $\mathcal{H}_0^{\gamma/2} := \mathcal{H}_0^{\gamma/2}([0,1])$. This is a Hilbert space whose elements coincide a.e. with continuous functions vanishing at 0 and 1. On $\mathcal{H}_0^{\gamma/2}$, the two norms $\|\cdot\|_{\mathcal{H}^{\gamma/2}}$ and $\|\cdot\|_{\gamma/2}$ are equivalent.

We now extend the definition of the regional fractional Laplacian on $(0,1)$, which has been defined on $C^{\infty}((0,1))$, to the space $\mathcal{H}^{\gamma/2}$.

Definition 2 *For $\rho \in \mathcal{H}^{\gamma/2}$ we define the distribution $\mathbb{L}\rho$ by*

$$\langle \mathbb{L}\rho, G\rangle = \langle \rho, \mathbb{L}G\rangle, \quad G \in C_c^{\infty}((0,1)).$$

Let us check that $\mathbb{L}\rho$ is indeed a well defined distribution. Consider a sequence $\{G_n\}_{n\geq 1} \in C_c^{\infty}((0,1))$ converging to 0 in the usual topology of the test functions. By the integration by parts formula for the regional fractional Laplacian (see Theorem 3.3 in [5]) we have that for any $\rho \in \mathcal{H}^{\gamma/2}$, $\langle \mathbb{L}\rho, G_n\rangle = \langle \rho, G_n\rangle_{\gamma/2}$. Now using the Cauchy-Schwarz inequality and the mean value theorem, we get that $\langle \mathbb{L}\rho, G_n\rangle$ is bounded from above by a constant times

$$\|\rho\|_{\gamma/2}\|G_n\|_{\gamma/2} \lesssim \|\rho\|_{\gamma/2}\|G_n'\|_{\infty}^2 \iint_{[0,1]^2} |u-v|^{1-\gamma} du dv,$$

which goes to 0 as $n \to \infty$ since $\gamma \in (1,2)$. Therefore $\mathbb{L}\rho$ is a well defined distribution.

2.3 Hydrostatic equation

In this section we define the partial differential equation that the empirical density solves in the thermodynamic limit $N \to \infty$.

Then, we define the weak solutions of the partial differential equation which we will deal with. In order to state the hydrostatic limit and fractional Fick's law we first define the Hydrostatic equation.

Definition 3 *Let $\hat{\kappa} > 0$ be some parameter. We say that $\bar{\rho}^{\hat{\kappa}} : [0,1] \to [0,1]$ is a weak solution of the stationary regional fractional reaction-diffusion equation with non-homogeneous Dirichlet boundary conditions given by*

$$\begin{cases} \mathbb{L}_{\hat{\kappa}}\bar{\rho}^{\hat{\kappa}}(u) + \hat{\kappa}V_0(u) = 0, \quad u \in (0,1), \\ \bar{\rho}^{\hat{\kappa}}(0) = \alpha, \quad \bar{\rho}^{\hat{\kappa}}(1) = \beta, \end{cases} \tag{11}$$

if:

1. $\bar{\rho}^{\hat{\kappa}} \in \mathcal{H}^{\gamma/2}$.

2. $\displaystyle\int_0^1 \left\{ \frac{\left(\alpha - \bar{\rho}^{\hat{\kappa}}(u)\right)^2}{u^{\gamma}} + \frac{\left(\beta - \bar{\rho}^{\hat{\kappa}}(u)\right)^2}{(1-u)^{\gamma}} \right\} du < \infty.$

3. *For any function $G \in C_c^\infty((0,1))$ we have that*

$$\left\langle \bar{\rho}^{\hat{\kappa}}, \mathbb{L}_{\hat{\kappa}} G \right\rangle + \hat{\kappa} \langle G, V_0 \rangle = 0.$$

Remark 4 *The interior regularity of this solution is studied in [6], but the regularity at the boundary is unknown. In general, the regularity of $\bar{\rho}^\kappa$ at the boundaries is an open problem. We know about the regularity for some particular cases of κ. For example, for $\kappa = \infty$ we have an explicit expression given by $\bar{\rho}^\infty(u) = \dfrac{V_0(u)}{V_1(u)}$ for all $u \in [0,1]$, which has Hölder regularity equal to γ at the boundaries. For $\kappa = 1$, the profile $\bar{\rho}^1$ is given in terms of a Poisson kernel and it has Hölder regularity equal to $\frac{\gamma}{2}$ at the boundaries (see [3]).*

Remark 5 *In fact, item 1. and item 2. of the previous definition implies that $\rho^{\hat{\kappa}}(0) = \alpha$ and $\rho^{\hat{\kappa}}(1) = \beta$, we can see the proof of this in [2].*

Remark 6 *Since $\bar{\rho}^\infty$ is a continuous function such that*

$$\int_0^1 \left\{ \frac{(\alpha - \bar{\rho}^\infty(u))^2}{u^\gamma} + \frac{(\beta - \bar{\rho}^\infty(u))^2}{(1-u)^\gamma} \right\} du < \infty$$

and $\bar{\rho}^\infty(0) = \alpha$ and $\bar{\rho}^\infty(1) = \beta$. Moreover, we can use item 1. and item 2. in Definition 3 to obtain that $\bar{\rho}^\kappa - \bar{\rho}^\infty \in \mathcal{H}_0^{\gamma/2}([0,1]) \cup L^2_{V_1}([0,1])$.

Lemma 7 *There exists a unique weak solution of (11).*

Proof. See the proof in section 6.2 of [2]. ■

2.4 Statement of results

We study in this subsection the asymptotic behavior of the empirical measure under the stationary state $\bar{\mu}_N$ (hydrostatic limit) for the case where $\kappa \geq 0$. As a result of hydrostatic limit we obtain a fractional version of the Fick's law.

Theorem 8 *(Hydrostatic limit) Let $\gamma \in (1,2)$ and $\kappa \geq 0$. For any continuous function $G : [0,1] \to \mathbb{R}$ we have that*

$$\lim_{N\to\infty} \frac{1}{N-1} \sum_{z=1}^{N-1} G(\tfrac{z}{N}) \eta_z = \int_0^1 G(u) \bar{\rho}^\kappa(u) du$$

in probability under $\bar{\mu}_N$, where $\bar{\rho}^\kappa$:$\mathbb{R} \to [0,1]$ is the unique weak solution of (11).

The classic Fick's law describes diffusion phenomena. In the standard case, the diffusion turns out to be described locally. However, in this paper we are considering a model which presents a non-standard diffusion and it will not be described locally. Our second result is the following "fractional Fick's law". Recall the definition of the current W_x (see (3)).

Theorem 9 *(Fractional Fick's law) The following fractional Fick's law holds*

$$\begin{aligned}\lim_{N\to\infty} N^{\gamma-1}\langle W_{[uN]}\rangle_N = c_\gamma \int_0^u \int_u^1 \frac{\bar\rho^\kappa(v)-\bar\rho^\kappa(w)}{(w-v)^{1+\gamma}}dwdv \\ + \kappa\int_u^1 (\alpha-\rho^\kappa(v))r^-(v)dv \\ + \kappa\int_0^u (\beta-\rho^\kappa(v))r^+(v)dv,\end{aligned} \tag{12}$$

where $\bar\rho^\kappa : \mathbb{R} \to [0,1]$ *is the unique weak solution of (11) and* u *is arbitrary in* $(0,1)$.

We can observe that the current is a non-local function of the density. The right hand side of (12) does not depend on u. This can be proved by taking the derivative with respect to u on the right hand side of (12) and showing that it vanishes thanks to (3).

An important step in the proof of Theorem 9 is to use stationarity of $\bar\mu_N$ which give an upper bound of the average current.

Lemma 10 *Fix* $N \geq 2$. *There exists a constant* $C > 0$ *such that* $\langle W_1\rangle_N \leq CN^{1-\gamma}$.

The proofs of Theorem 9 and Lemma 10 are similar to the argument done in Theorem 2.4 and Lemma 4.1 of [3], for this reason we omit them.

3 Hydrostatic limit and Fick's law

In this section we prove Theorem 8. Let $\mathcal{M}_d^+$, $d = 1,2$, be the space of positive measures on $[0,1]^d$ with total mass bounded by 1 equipped with the weak topology. For any $\eta \in \Omega^N$ the empirical measures $\pi^N(\eta) \in \mathcal{M}_1^+$ (resp. $\hat\pi^N(\eta) \in \mathcal{M}_2^+$) are defined by

$$\pi^N(\eta) = \frac{1}{N-1}\sum_{x=1}^{N-1} \eta_x \delta_{x/N} \quad \Big(\text{resp. } \hat\pi^N(\eta) = \frac{1}{(N-1)^2}\sum_{x,y=1}^{N-1} \eta_x\eta_y\delta_{(x/N,y/N)}\Big)$$

where δ_u (resp. $\delta_{(u,v)}$) is the Dirac mass on $u \in [0,1]$ (resp. $(u,v) \in [0,1]^2$). Let $\mathbb{P}^N$ be the law on $\mathcal{M}_1^+ \times \mathcal{M}_2^+$ induced by $(\pi^N, \hat{\pi}^N) : \Omega^N \to \mathcal{M}_1^+ \times \mathcal{M}_2^+$ when Ω^N is equipped with the non-equilibrium stationary state $\bar{\mu}_N$. To simplify notations, we denote $\pi^N(\eta)$ (resp. $\hat{\pi}^N(\eta)$) by π^N (resp. $\hat{\pi}^N$) and the action of $\pi \in \mathcal{M}_d^+$ on a continuous function $G : [0,1]^d \to \mathbb{R}$ by $\langle \pi, G \rangle = \int_{[0,1]^d} G(u)\pi(du)$.

Our goal is to prove that every limit point $\mathbb{P}^*$ of the sequence $\{\mathbb{P}^N\}_{N\geq 2}$ is concentrated on the set of measures $(\pi, \hat{\pi})$ of $\mathcal{M}_1^+ \times \mathcal{M}_2^+$ such that π (resp. $\hat{\pi}$) is absolutely continuous with respect to the Lebesgue measure on $[0,1]$ (resp. $[0,1]^2$) and whose density ρ^κ (resp. $\bar{\rho}^\kappa(u)\bar{\rho}^\kappa(v)$) is a weak solution of (11).

Lemma 11 *The sequence $\{\mathbb{P}^N\}_{N\geq 2}$ is tight. Let $\mathbb{P}^*$ be a limit point of the sequence $\{\mathbb{P}^N\}_{N\geq 2}$. Then $\mathbb{P}^*$ is concentrated on absolutely continuous measures $(\pi(du), \hat{\pi}(dudv)) = (\pi(u)du, \pi(u)\pi(v)dudv)$. The density π is a positive function in $\mathcal{H}^{\gamma/2}([0,1])$ and satisfies $\int_0^1 \left\{ \frac{(\alpha-\pi(u))^2}{u^\gamma} + \frac{(\beta-\pi(u))^2}{(1-u)^\gamma} \right\} du < \infty$.*

Proof. Since $\mathcal{M}_d^+$ is compact in the weak topology we have that the sequence $\{\mathbb{P}^N\}_{N\geq 2}$ is tight on $\mathcal{M}_d^+$ (see e.g [4]). $\mathbb{P}^*$ is concentrated on absolutely continuous measures because the process allows one particle per site. Since $\hat{\pi}^N$ is a product measure whose marginals are given by π^N, by weak convergence, we have that $\hat{\pi}(u,v) = \pi(u)\pi(v)$ for any $(u,v) \in [0,1]^2$.

The proof that the density $\pi \in \mathcal{H}^{\gamma/2}([0,1])$ and satisfies

$$\int_0^1 \left\{ \frac{(\alpha - \pi(u))^2}{u^\gamma} + \frac{(\beta - \pi(u))^2}{(1-u)^\gamma} \right\} du < \infty,$$

is similar to the one done in Theorem 3.6 in [2] and the fact that $\bar{\mu}_N$ is stationary measure. ■

Let $\mathbb{P}^*$ be a limit point of the sequence $\{\mathbb{P}^N\}_{N\geq 2}$ whose existence follows from the previous Lemma. Hereinafter, we assume without lost of generality that $\{\mathbb{P}^N\}_{N\geq 2}$ converges to $\mathbb{P}^*$.

Lemma 12 *Let $\bar{\rho}^\kappa$ be the unique weak solution of (11). For any F, G in $C_c^\infty([0,1])$ we have*

$$\int_{[0,1]^2} \{F(u)(\mathbb{L}_\kappa G)(v) + G(v)(\mathbb{L}_\kappa F)(u)\} I^\kappa(u,v)dudv = 0, \tag{13}$$

where

$$I^\kappa(u,v) = \mathbb{E}^* \left[(\pi(u) - \bar{\rho}^\kappa(u)) (\pi(v) - \bar{\rho}^\kappa(v)) \right]. \tag{14}$$

Proof. We have that

$$
\begin{aligned}
N^\gamma L_N(\langle \pi^N, G\rangle) =& \frac{1}{1-N}\sum_{x\in\Lambda_N}\left[N^\gamma \sum_{y\in\Lambda_N}\left(G(\tfrac{y}{N})-G(\tfrac{x}{N})\right)p(y-x)\right]\eta_x \\
&+\frac{\kappa N^\gamma}{N-1}\sum_{x\in\Lambda_N} G(\tfrac{x}{N})\left[r_N^-(\tfrac{x}{N})(\alpha-\eta_x)+r_N^+(\tfrac{x}{N})(\beta-\eta_x)\right].
\end{aligned}
\tag{15}
$$

Taking the expectation with respect to $\bar{\mu}_N$ on both sides of (15), by stationarity the left hand side vanishes. By using Lemma 3.3 in [3] and weak convergence we have that

$$
\mathbb{E}^*\left[\int_0^1 (\mathbb{L}_\kappa G)(u)\pi(u)du\right]+\kappa\int_0^1 V_0(u)G(u)du=0. \tag{16}
$$

By a similar argument done in Lemma 4.6 in [3] we get

$$
\begin{aligned}
&\mathbb{E}^*\left[\int_{[0,1]^2} F(u)(\mathbb{L}_\kappa G)(v)\pi(u)\pi(v)dudv\right] \\
+&\mathbb{E}^*\left[\int_{[0,1]^2} G(v)(\mathbb{L}_\kappa F)(u)\pi(u)\pi(v)dudv\right] \\
-&\mathbb{E}^*\left[\kappa\int_{[0,1]^2}\{F(u)G(v)V_0(v)\,\pi(u)+F(u)G(v)V_0(u)\pi(v)\}\,dudv\right]=0.
\end{aligned}
\tag{17}
$$

Let $\bar{\rho}^\kappa$ be the unique weak solution of (11). Then we have

$$
\int_0^1 (\mathbb{L}_\kappa G)(u)\bar{\rho}^\kappa(u)du+\kappa\int_0^1 G(u)V_0(u)\,du=0, \tag{18}
$$

for all $G\in C_c^\infty((0,1))$. By using (16) we can get that

$$
\begin{aligned}
&\mathbb{E}^*\left[\int_{[0,1]^2} F(u)(\mathbb{L}_\kappa G)(v)\pi(v)\bar{\rho}^\kappa(u)dudv\right] \\
&+\kappa\int_{[0,1]^2} F(u)G(v)V_0(v)\,\bar{\rho}^\kappa(u)dudv=0
\end{aligned}
\tag{19}
$$

and

$$\begin{aligned} &\mathbb{E}^*\left[\int_{[0,1]^2} G(v)(\mathbb{L}_\kappa F)(u)\pi(u)\bar{\rho}^\kappa(v)dudv\right] \\ &+\kappa\int_{[0,1]^2} F(u)G(v)V_0(u)\,\bar{\rho}^\kappa(v)dudv = 0. \end{aligned} \tag{20}$$

Now, from (18) we can get the following equations

$$\begin{aligned} &\mathbb{E}^*\left[\int_{[0,1]^2} F(u)(\mathbb{L}_\kappa G)(v)\pi(u)\bar{\rho}^\kappa(v)dudv\right] \\ &+\mathbb{E}^*\left[\kappa\int_{[0,1]^2} F(u)G(v)V_0(v)\,\pi(u)dudv\right] = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} &\mathbb{E}^*\left[\int_{[0,1]^2} G(v)(\mathbb{L}_\kappa F)(u)\pi(v)\bar{\rho}^\kappa(u)dudv\right] \\ &+\mathbb{E}^*\left[\kappa\int_{[0,1]^2} F(u)G(v)V_0(u)\,\pi(v)dudv\right] = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} &-\int_{[0,1]^2} F(u)(\mathbb{L}_\kappa G)(v)\bar{\rho}^\kappa(v)\bar{\rho}^\kappa(u)dudv \\ &-\kappa\int_{[0,1]^2} F(u)G(v)V_0(v)\,\bar{\rho}^\kappa(u)dudv = 0 \end{aligned} \tag{23}$$

and

$$\begin{aligned} &-\int_{[0,1]^2} G(v)(\mathbb{L}_\kappa G)(u)\bar{\rho}^\kappa(u)\bar{\rho}^\kappa(v)dudv \\ &-\kappa\int_{[0,1]^2} F(u)G(v)V_0(u)\,\bar{\rho}^\kappa(v)dudv = 0. \end{aligned} \tag{24}$$

Using equations (17), (19)-(24), then it follows (13). ■

Now we define the operator

$$(\mathbb{L}_1 H)(u,v) := (\mathbb{L}H(\cdot,v))(u)$$

with $H \in C^\infty((0,1)^2)$. Similarly we define $\mathbb{L}_2$ by acting over the second coordinate. We define the operator $\mathcal{L} = \mathbb{L}_1 + \mathbb{L}_2$. Note that if $H(u,v) = F(u)G(v)$ then we get that

$$(\mathcal{L}H)(u,v) = F(u)(\mathbb{L}G)(v) + G(v)(\mathbb{L}F)(u).$$

We also define the semi inner-product $\langle \cdot, \cdot \rangle_{1,\gamma/2}$ on $G \in C^\infty((0,1)^2)$ as

$$\langle F, G \rangle_{1,\gamma/2} = \int_0^1 \langle F(\cdot, v), G(\cdot, v) \rangle_{\gamma/2} dv,$$

and its corresponding semi-norm is denoted by $\| \cdot \|_{1,\gamma/2}$. Similarly we define $\langle \cdot, \cdot \rangle_{2,\gamma/2}$ and $\| \cdot \|_{2,\gamma/2}$.

We also consider the space $\mathbb{H}^{\gamma/2}_{0,\hat{V}} := \mathbb{H}^{\gamma/2}_{0,\hat{V}}([0,1]^2)$ as the set of integrable functions $H : (0,1)^2 \to \mathbb{R}$ vanishing at the boundary and such that

$$\|H\|^2_{\mathbb{H}^{\gamma/2}_{0,\hat{V}}} := \|H\|^2_{1,\gamma/2} + \|H\|^2_{2,\gamma/2} + \|H\|^2_{\hat{V}} < \infty,$$

where $\hat{V}(u,v) = V_1(u) + V_1(v)$. Let us consider the following definition needed in the proof of Theorem 8.

Definition 13 *We say that $\bar{I}^\kappa : [0,1]^2 \to [0,1]$ is a weak solution of*

$$\begin{cases} (\mathcal{L}\bar{I}^\kappa)(u,v) - \kappa \bar{I}^\kappa(u,v)\hat{V}(u,v) = 0, & (u,v) \in (0,1)^2, \\ \bar{I}^\kappa(u,v) = 0, & (u,v) \in \partial[0,1]^2. \end{cases} \tag{25}$$

if

1. $\bar{I}^\kappa \in \mathbb{H}^{\gamma/2}_{0,\hat{V}}$.
2. *For any function $H \in C_c^\infty((0,1)^2)$ we have that*

$$\langle \bar{I}^\kappa, \hat{\mathcal{L}} H \rangle - \kappa \langle \bar{I}^\kappa, H \rangle_{\hat{V}} = 0. \tag{26}$$

Lemma 14 *The unique weak solution of (25) is the constant function equal to zero.*

Proof. It is clear that a zero function is a weak solution of (25). Now, we use Lax-Milgram's theorem in order to prove the uniqueness.

Let $B^\kappa : \mathbb{H}^{\gamma/2}_{0,\hat{V}}([0,1]^2) \times \mathbb{H}^{\gamma/2}_{0,\hat{V}}([0,1]^2) \to \mathbb{R}$ a bilinear form defined as

$$B^\kappa(\varphi, \varrho) = \langle \varphi, \varrho \rangle_{1,\gamma/2} + \langle \varphi, \varrho \rangle_{2,\gamma/2} + \kappa \langle \varphi, \varrho \rangle_{\hat{V}},$$

for any functions $\varphi, \varrho \in \mathbb{H}^{\gamma/2}_{0,\hat{V}}$. We note that B^κ is coercive, indeed

$$B^\kappa(\varphi, \varphi) = \|\varphi\|^2_{1,\gamma/2} + \|\varphi\|^2_{2,\gamma/2} + \kappa \|\varphi\|^2_{\hat{V}} \geq \min\{1, \kappa\} \|\varphi\|^2_{\mathbb{H}^{\gamma/2}_{0,\hat{V}}}. \tag{27}$$

Now by using Cauchy-Schwarz inequality we can get that

$$|B^\kappa(\varphi,\varrho)| \le \|\varphi\|_{1,\gamma/2}\|\varrho\|_{1,\gamma/2} + \|\varphi\|_{2,\gamma/2}\|\varrho\|_{2,\gamma/2} + \kappa\|\varphi\|_{\hat{V}}\|\varrho\|_{\hat{V}}.$$

This inequality allows us to conclude that the bilinear form B^κ is also continuous. Then Lax-Milgram's theorem guarantees that there exists a unique function $\bar{I}^\kappa$, which satisfies (26) for any function $H \in C_c^\infty((0,1)^2)$. ■

3.1 Proof of theorem 8

Let $\bar{\rho}^\kappa(u)$ the unique weak solution of (11) and recall the definition of the function $I^\kappa : [0,1]^2 \to \mathbb{R}$ introduced in Lemma 12. We want to prove that I^κ is a weak solution of (25). First, we claim that $I^\kappa \in \mathbb{H}^{\gamma/2}_{0,\hat{V}}$. Indeed, since $\bar{\rho}^\kappa, \pi \in \mathcal{H}^{\gamma/2}$ (see Definition 3 and Lemma 11) then we have that $(\pi - \bar{\rho}^\kappa) \in \mathcal{H}^{\gamma/2}_0$ and $\|I^\kappa\|_{i,\gamma/2}$ are finite for $i = 1,2$. In order to show that $I^\kappa \in L^2_{\hat{V}}$, note that $\int_{[0,1]^2}(I^\kappa(u,v))^2\hat{V}(u,v)dudv$ is less than

$$\mathbb{E}^*\left[\int_{[0,1]^2} P^2(u,v)\hat{V}(u,v)dudv\right] \le 2\mathbb{E}^*\left[\int_{[0,1]^2} P^2(u,v)V_1(v)dudv\right], \quad (28)$$

where $P(u,v) = (\pi(u) - \bar{\rho}^\kappa(u))(\pi(v) - \bar{\rho}^\kappa(v))$ and in the last inequality we performed a change of variables. Also, we can observe the term on the right hand side of (28) is bounded from above by

$$4\mathbb{E}^*\left[\int_0^1 (\pi(u) - \bar{\rho}^\kappa(u))^2 du \int_0^1 \left((\pi(v) - \bar{\rho}^\infty(v))^2 + (\bar{\rho}^\infty(v) - \bar{\rho}^\kappa(v))^2\right) V_1(v)dv\right]. \quad (29)$$

We know that $\pi, \bar{\rho}^\kappa$ satisfy items 1. and 2. then by Remarks 5 and 6 we have that (29) is finite. Therefore we get that $I^\kappa \in L^2_{\hat{V}}$. Now, by Lemma 12 we have that the function I^κ is a weak solution of (25) (note that in Definition 13 the test function can be taken as the product of two test functions on $C_c^\infty((0,1))$). By Lemma 14 we have that $I^\kappa \equiv 0$. Whence we conclude that $I^\kappa(u,u) = 0$ for all $u \in (0,1)$ or equivalently $\mathbb{P}^*$ almost surely $\pi = \bar{\rho}^\kappa$. This conclude the proof of Theorem 8.

Acknowledgements

The author B. Jiménez is very grateful to Cédric Bernardin and Patrícia Gonçalves for useful teachings and discussions.

Funding

The work was partially funded by the UNA National University and by the authors in their free time.

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