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NOTES ON COHERENT SYSTEMS

NOTAS SOBRE LOS SISTEMAS COHERENTES

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Abstract

We present an alternative approach to semistability and moduli spaces for coherent systems associated with decorated vector bundles. In this approach, it seems possible to construct a Hitchin map. We relate some examples to classical problems from geometric invariant theory.

Keywords: coherent system; moduli space; Hitchin map; first fundamental theorem of invariant theory.

Resumen

En estas notas se presenta un nuevo enfoque para el estudio de las condiciones de semi-estabilidad, así como de los espacios de móduli, de los sistemas coherentes asociados a fibrados vectoriales con estructura adicional. Bajo este enfoque, se abre la posibilidad de definir un morfismo de Hitchin. Se muestra, además, la relación entre algunos ejemplos concretos con problemas clásicos presentes en la teoría geométrica de invariantes.

Palabras clave: sistema coherente; espacio de móduli; morfismo de Hitchin; primer teorema fundamental de la teoría de invariantes.

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1 Introduction

The group $G := \mathrm{GL}_r(\mathbb{C})$ is the group of linear automorphisms of the complex vector space $Q := \mathbb{C}^r$. So, tautologically, there is the action

$$\begin{aligned} G \times Q &\longrightarrow Q \\ (g, x) &\longmapsto g \cdot x. \end{aligned}$$

There are many actions which may be derived from it, e.g.,

1. the action

$$\begin{aligned} G \times \mathrm{Mat}_r(\mathbb{C}) &\longrightarrow \mathrm{Mat}_r(\mathbb{C}) \\ (g, m) &\longmapsto g \cdot m \cdot g^{-1} \end{aligned}$$

   of G on the vector space of $(r \times r)$-matrices,

2. the action

$$\begin{aligned} G \times \mathrm{Mat}_r(\mathbb{C})^{\oplus 2} &\longrightarrow \mathrm{Mat}_r(\mathbb{C}) \\ (g, m_1, m_2) &\longmapsto (g \cdot m_1 \cdot g^{-1}, g \cdot m_2 \cdot g^{-1}) \end{aligned}$$

   of G on the vector space of pairs of $(r \times r)$-matrices,

3. for $1 \leq s \leq r-1$, the action

$$\begin{aligned} G \times \mathrm{Gr}_s(Q) &\longrightarrow \mathrm{Gr}_s(Q) \\ (g, R) &\longmapsto g \cdot R \end{aligned}$$

of G on the Graßmannian variety $\mathrm{Gr}_s(Q)$ of s-dimensional sub vector spaces of Q, and

4. for $a \geq 2$ and $1 \leq s \leq a \cdot r - 1$, the action

$$\begin{aligned} G \times \mathrm{Gr}_s(Q^{\otimes a}) &\longrightarrow \mathrm{Gr}_s(Q^{\otimes a}) \\ (g, R) &\longmapsto g \cdot R \end{aligned}$$

of G on the Graßmannian variety $\mathrm{Gr}_s(Q^{\otimes a})$ of s-dimensional sub vector spaces of $Q^{\otimes a}$. Here, G acts on $Q^{\otimes a}$ by the a-fold tensor power of the initial "tautological" action of G on Q.

The task would be to describe the set of G-orbits in the respective space. For the original action, there are two orbits, namely $\{0\}$ and $Q \setminus \{0\}$. In Example (3), there is only one orbit. For Example (1), there is, e.g., the theory of the Jordan normal form which exhibits in each orbit a "nice" representative. However, Case (2) has not been solved, so far, and is, in fact, considered a wild problem which one does not expect to solve completely. The same goes for Case (4). In order to say something meaningful in those cases, we need to endow the set of orbits with an additional structure. The easiest structure is that of a topological space — simply endow the set of orbits with the quotient topology. As algebraic geometers, we prefer the structure of an algebraic variety. If the set of G-orbits carries the structure of an algebraic variety, the topological space described before needs to be a Hausdorff space. If there are non-closed orbits, as in Example (1) for $r \geq 2$, this is not true. So, we may hope at best to equip the set of closed orbits with the structure of an algebraic variety. In the case that G acts on an affine variety, such as in Example (1) and (2), Hilbert's seminal work [10] and [11], as reinterpreted by Mumford [22], shows that the set of closed orbits carries a natural structure of an affine algebraic variety. That variety is called the *categorical quotient*. If G acts on a projective variety as in Example (3) and (4), then one needs an additional datum, called *linearization*, in order to find a G-invariant open subset U, such that the set of closed orbits inside U carries the structure of a projective variety and yields the categorical quotient (see [22]).

Before we proceed, let us mention some related actions which occur in applications. First, we look at the diagram

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n$$

and pick positive integers $r_1, ..., r_n$. Then, there is the action

$$\begin{aligned} \bigtimes_{i=1}^{n} \mathrm{GL}_{r_i}(\mathbb{C}) \times \bigoplus_{i=1}^{n-1} \mathrm{Mat}_{r_{i+1},r_i}(\mathbb{C}) &\longrightarrow \bigoplus_{i=1}^{n-1} \mathrm{Mat}_{r_{i+1},r_i}(\mathbb{C}) \\ (g_1, ..., g_n, m_1, ..., m_{n-1}) &\longmapsto (g_2 \cdot m_1 \cdot g_1^{-1}, ..., g_n \cdot m_{n-1} \cdot g_{n-1}^{-1}). \end{aligned}$$

In this case, there are only finitely many orbits, and representatives are direct sums of the basic diagrams

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{C} \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} \mathbb{C} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0.$$

This allows to classify orbits by bar codes and plays an important rôle in persistent homology and big data analysis [24]. More generally, for each directed graph and choice of a positive integer for each vertex, one has a similar action. There are either finitely many orbits, or the classification problem is as complex as the one for Example (1), or the problem is as wild as Example (2). (We refer to [24], Appendix A, for a guide to the literature.) Some of the latter diagrams are considered in control theory. The choice of a linearization is important for obtaining compactifications of moduli spaces of linear systems. We refer to [9] and [1] for overviews and general results. The paper [12] explains the rôle of Example (1) in control theory more explicitly.

Now, we work relative to a smooth projective algebraic curve X over the complex numbers or a compact Riemann surface. Given an action

$$\sigma \colon G \times V \longrightarrow V$$

of $G = \mathrm{GL}_r(\mathbb{C})$ on a finite dimensional complex vector space V, corresponding to the group homomorphism

$$\begin{aligned} \varrho \colon G &\longrightarrow \mathrm{GL}(V) \\ g &\longmapsto \big(v \longmapsto \sigma(g, v)\big), \end{aligned}$$

called a *representation*, it is possible to associate with every algebraic or holomorphic vector bundle E of rank r on X an algebraic or holomorphic vector bundle E_ϱ on X with typical fiber V. In addition, we fix an algebraic or holomorphic line bundle L on X. The task, here, is to classify pairs (E, s) which consist of an algebraic or holomorphic vector bundle E on X and a section $s \colon \mathcal{O}_X \longrightarrow E_\varrho \otimes L$. (In fact, we may consider this problem over any algebraic or complex analytic variety X. The case that X is a point is the one that we discussed at the beginning.) For the first action which we considered and $L = \mathcal{O}_X$, we look at vector bundles E together with a global section $s \in H^0(X, E)$. These are the Bradlow pairs introduced in [3]. For Example (1) and $L = \omega_X$, the cotangent bundle of X, we have vector bundles E together with a twisted endomorphism $s\colon E \longrightarrow E \otimes \omega_X$. These are known as Higgs bundles (see [13]). Building on these and many more examples, the author developed a procedure for constructing moduli spaces for these objects (see [32]). These moduli spaces are the analogs to the categorical quotients discussed above. The differential geometric counterparts of these objects occur in mathematical physics in the context of gauge theory ([35], Chapter V, Section 4-6). There is a so-called Kobayashi–Hitchin correspondence between related objects in gauge theory and semistable decorated vector bundles in the above sense ([21], [20]). The theory of decorated vector bundles that we have just outlined is closely related to the theory of stable gauged maps which has been used in quantum cohomology. We refer to [7] for an introduction.

The example of the action of G on the Graßmannian has been generalized to the moduli problem of Brill–Noether pairs or coherent systems. These are pairs (E, Γ) in which E is a vector bundle of rank r and $\Gamma \subset H^0(X, E)$ is a subspace of prescribed dimension. Raghavendra and Vishwanath [26] formulated a notion of semistability for coherent systems which depends on a parameter $\delta \in \mathbb{Q}_{>0}$ and constructed the moduli spaces for some values of δ. Constructions of moduli spaces which work in general were given by King and Newstead [15] and Le Potier [19]. In [34], the author considered the corresponding analog for the action of G on $\mathrm{Gr}_s(Q^{\otimes a})$. To this end, we fix again a line bundle L on X. Then, a coherent system is a pair (E, Γ) which consists of a vector bundle E of rank r on X and a vector space $\Gamma \subset H^0(X, E^{\otimes a} \otimes L)$. The paper [34] introduces a notion of stability for these coherent systems which generalizes the notion of Raghavendra and Vishwanath and seems quite natural. This notion also depends on a stability parameter $\delta \in \mathbb{Q}_{>0}$. The paper also contains a construction of moduli spaces. However, the construction works only for stability parameters below a certain threshold. In the explicit study of moduli spaces, the variation of the moduli spaces with the stability parameter δ plays an important rôle and the moduli spaces for large values of δ are typically easier to understand. We refer to [39] and [4] for two characteristic examples. Motivated by this, we look in this note at another possible notion of semistability which also depends on a positive rational number δ. It is more complicated to state, but the construction of moduli spaces we propose does not impose an upper bound on the stability parameter. This approach follows the natural idea to construct the moduli spaces for coherent systems as $\mathrm{GL}_s(\mathbb{C})$-quotients of the moduli spaces for decorated vector bundles associated with $\varrho^{\oplus s}$, $\varrho\colon \mathrm{GL}_r(\mathbb{C}) \longrightarrow \mathrm{GL}(\mathbb{C}^{r\otimes a})$ being the a-

fold tensor power of the standard representation and $s := \dim_{\mathbb{C}}(\Gamma)$. In addition, it seems that, for some of the moduli spaces, there exists a Hitchin map. Besides the definition of semistability and a sketch of the construction of moduli spaces, this note explores some examples which should help to understand the general picture, in particular, the Hitchin map. In future work, we hope to address the relation between the constructions of [34] and this note in order to fully understand the variation of moduli spaces. We would also like to point out that twisted versions of Brill–Noether pairs appear, e.g., in [38] and [14]. One fixes a vector bundle V on X and looks at pairs (E, Γ) where E is a vector bundle on X and $\Gamma \subset H^0(X, E \otimes V)$ is a linear subspace. I was informed by Peter Newstead that the case when both E and V vary is also of interest for Brill–Noether theory. This motivates to develop the results of [34] and this note further to a theory of coherent systems for decorated principal bundles. The case of Brill–Noether pairs with varying E and V would correspond to the structure group $\mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_s(\mathbb{C})$ and the natural representation of this group on $\mathbb{C}^r \otimes \mathbb{C}^s$.

Notation

We will work on a connected smooth projective curve X of genus g at least two which is defined over the field $\mathbb{C}$ of complex numbers, and we will fix a point $x_0 \in X$. We write $\mathcal{O}_X(k)$ for $\mathcal{O}_X(k \cdot x_0)$, and, given a coherent $\mathcal{O}_X$-module $\mathcal{F}$, the symbol $\mathcal{F}(k)$ stands for the $\mathcal{O}_X$-module $\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(k)$, $k \in \mathbb{Z}$.

Given a scheme S and a vector bundle A on S, we write $P(A)$ for the projective bundle of lines in the fibers of A, i.e., for $\underline{\mathrm{Proj}}(\underline{\mathrm{Sym}}^\star(A^\vee))$. Occasionally, we will also use $\mathbb{P}(A) := P(A^\vee)$, i.e., Grothendieck's convention for the projectivization.

For a cartesian product $A \times B$ in a category, we let $\pi_A \colon A \times B \longrightarrow A$ and $\pi_B \colon A \times B \longrightarrow B$ be the natural projections.

2 Review of some moduli spaces of decorated bundles

In [6], a general theory for tuples of vector bundles on a smooth projective curve X decorated with the help of a multihomogeneous representation was developed. It includes the classification of triples (F, E, φ), consisting of a vector bundle F of rank s, a vector bundle E of rank r, and a homomorphism $\varphi \colon F \longrightarrow E^{\otimes a} \otimes L$, for a fixed line bundle L on X. The case $F \cong \mathcal{O}_X^{\oplus s}$ corresponds to coherent systems. We will first review the relevant results from the theory of decorated vector bundles from [30] and then present a moduli problem for decorated pairs of vector bundles. It will be important to understand some features of that moduli problem in order to understand in which respects the moduli problem for coherent systems is different.

2.1 Decorated vector bundles

Let a, s, and r be positive integers. Suppose that $\varrho\colon \mathrm{GL}_r(\mathbb{C}) \longrightarrow \mathrm{GL}(V)$ is a representation which is homogeneous of degree a, e.g., the a-th tensor power of the standard representation with $V = (\mathbb{C}^r)^{\otimes a}$.

Then, given a vector bundle E of rank r on X, we may use ϱ to assign to E a vector bundle E_ϱ of rank $\dim_{\mathbb{C}}(V)$. Fix also a line bundle L on X. A *ϱ-pair* is a pair (E, φ) which consists of a vector bundle E of rank r on X and a non-zero homomorphism $\varphi\colon \mathcal{O}_X^{\oplus s} \longrightarrow E_\varrho \otimes L$.

Recall that a *weighted filtration* of a vector bundle E on X is a pair $(E_\bullet, \alpha_\bullet)$ in which

$$0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_a \subsetneq E$$

is a filtration of E by subbundles and $\alpha_\bullet = (\alpha_1, ..., \alpha_a)$ is a vector of positive rational numbers. For such a weighted filtration $(E_\bullet, \alpha_\bullet)$, we set

$$M(E_\bullet, \alpha_\bullet) = \sum_{i=1}^{a} \alpha_i \cdot \big(\deg(E) \cdot \mathrm{rk}(E_i) - \deg(E_i) \cdot \mathrm{rk}(E)\big).$$

Given a homomorphism $\varphi\colon \mathcal{O}_X^{\oplus s} \longrightarrow E_\varrho \otimes L$ and a weighted filtration $(E_\bullet, \alpha_\bullet)$ of E, the quantity $\mu(E_\bullet, \alpha_\bullet; \varphi)$ is defined in [30], p. 175f, [32], p. 139.

Let δ be a positive rational number. A ϱ-pair (E, φ) is *δ-(semi)stable*, if the inequality

$$M(E_\bullet, \alpha_\bullet) + \delta \cdot \mu(E_\bullet, \alpha_\bullet; \varphi) (\geq) 0$$

is satisfied, for every weighted filtration $(E_\bullet, \alpha_\bullet)$ of E.

For the rest of this section, we fix a line bundle M on X and look only at ϱ-pairs (E, φ) with $\det(E) \cong M$. Recall from [30], Theorem 2.3.7.1, that there are critical values $0 =: c_0 < c_1 < \cdots < c_t < c_{t+1} := \infty$, such that the notions of δ-semistability and δ-stability for ϱ-pairs (E, φ) with $\det(E) \cong M$ remain constant within each of the intervals $(c_i, c_{i+1}) \cap \mathbb{Q}$, $i = 0, ..., t$. A ϱ-pair (E, φ) is *asymptotically (semi)stable*, if it is δ-(semi)stable with respect to a stability parameter $\delta \in (c_t, \infty) \cap \mathbb{Q}$.

Lemma 2.1 *A ϱ-pair (E, φ) is asymptotically (semi)stable if and only if it satisfies the following two conditions.*

a) $\mu(E_\bullet, \alpha_\bullet; \varphi) \geq 0$, *for every weighted filtration* $(E_\bullet, \alpha_\bullet)$ *of* E,

b) $M(E_\bullet, \alpha_\bullet)(\geq)0$, *for every weighted filtration* $(E_\bullet, \alpha_\bullet)$ *of* E *with* $\mu(E_\bullet, \alpha_\bullet; \varphi) = 0$.

Remark 2.2 *Let η be the generic point of X, $\mathbb{K} := \mathbb{C}(X)$ the function field of the curve X, $\mathbb{E}$ the restriction of E to η, and $\mathbb{V}$ the restriction of E_ϱ to η. Note that $\mathbb{E}$ and $\mathbb{V}$ are vector spaces over the field $\mathbb{K}$. The restriction of φ to η yields a point $\varphi_\eta \in \operatorname{Hom}(\mathbb{K}^s, \mathbb{V})$. Since we are working in characteristic zero, the semistability of φ_η may be tested with the Hilbert–Mumford criterion (see [32], Section 1.7.1, for a brief discussion). Therefore, φ_η is semistable if and only if Condition a) in Lemma 2.1 is satisfied. In this case, we say that φ is generically semistable.*

For a positive rational number δ, we let $\mathcal{P}^\delta_{X/M/L/\varrho}$ be the moduli space of δ-semistable ϱ-pairs (E, φ) in which $\det(E) \cong M$. (We will briefly review the construction in Section 4.1.) For $\delta \in (c_t, \infty)$, the above moduli space is equipped with a special gadget, the so-called Hitchin map. To this end, we set $\mathbb{H} := \operatorname{Hom}(\mathbb{C}^s, V)$. The coordinate algebra of $\mathbb{H}$ is isomorphic to the symmetric algebra of $\mathbb{H}^\vee$,

$$\mathbb{C}[\mathbb{H}] \cong \operatorname{Sym}^\star(\mathbb{H}^\vee) = \bigoplus_{d \geq 0} \operatorname{Sym}^d(\mathbb{H}^\vee),$$

and the action of $\mathrm{SL}_r(\mathbb{C})$ preserves this grading, so that, in particular,

$$\mathbb{C}[\mathbb{H}]^{\mathrm{SL}_r(\mathbb{C})} = \bigoplus_{d \geq 0} \operatorname{Sym}^d(\mathbb{H}^\vee)^{\mathrm{SL}_r(\mathbb{C})}.$$

Note that the GIT quotient of $P(\mathbb{H})$ by the action of $\mathrm{SL}_r(\mathbb{C})$ is given as

$$P(\mathbb{H}) /\!\!/ \mathrm{SL}_r(\mathbb{C}) := \operatorname{Proj}(\mathbb{C}[\mathbb{H}]^{\mathrm{SL}_r(\mathbb{C})}).$$

For any $k > 0$, we set

$$\mathbf{sym}^\star_k := \bigoplus_{d \geq 0} \mathbf{sym}^d_k, \quad \mathbf{sym}^d_k := \operatorname{Sym}^{k \cdot d}(\mathbb{H}^\vee)^{\mathrm{SL}_r(\mathbb{C})}, \quad d \geq 0.$$

Then,

$$P(\mathbb{H}) /\!\!/ \mathrm{SL}_r(\mathbb{C}) \cong \operatorname{Proj}(\mathbf{sym}^\star_k), \quad k > 1.$$

We may choose $k > 0$ in such a way that $\mathbf{sym}_k^\star$ is generated by $\mathbf{sym}_k^1$, i.e,

$$P(\mathbb{H})/\!\!/\operatorname{SL}_r(\mathbb{C}) \hookrightarrow \mathbb{P}(\mathbf{sym}_k^1).$$

Remark 2.3 *A function $I \in \mathbb{C}[\mathbb{H}]$ is a ϱ-semiinvariant*, if there exists an integer w, such that

$$\forall g \in \operatorname{GL}_r(\mathbb{C}) \forall h \in \mathbb{H}: \quad I(g \cdot h) = \det(g)^w \cdot I(h).$$

The ring $\mathbb{C}[\mathbb{H}]^{\operatorname{SL}_r(\mathbb{C})}$ agrees with the ring of ϱ-semiinvariants. For a homogeneous element $I \in \mathbb{C}[\mathbb{H}]^{\operatorname{SL}_r(\mathbb{C})}$, the number $a \cdot \deg(I)$ is a multiple of r. In particular, we may write $a \cdot k = r \cdot l$.

Set

$$\mathbf{h}_1 := \mathbb{P}\big(\mathbf{sym}_k^1 \otimes H^0(X, M^{\otimes l} \otimes L^{\otimes k})\big). \tag{2.3.1}$$

Let $I_1, ..., I_u$ be a basis for $\mathbf{sym}_k^1$ and $\varphi\colon \mathcal{O}_X^{\oplus s} \longrightarrow E_\varrho \otimes L$ a homomorphism. We pick a suitable open covering $(U_i)_{i \in I}$ and trivializations $E_{\varrho|U_i} \cong V \otimes \mathcal{O}_{U_i}$, $L_{|U_i} \cong \mathcal{O}_{U_i}$, $i \in I$. So, φ defines morphisms $f_i\colon U_i \longrightarrow \mathbb{H}$ and, thus, functions $I_j \circ f_i$, $i \in I$. For $j \in \{1, ..., u\}$, the functions $I_j \circ f_i$, $i \in I$, glue to a section

$$I_j(\varphi) \in H^0(X, M^{\otimes l} \otimes L^{\otimes k}).$$

If φ is generically semistable, then there exists an index $j \in \{1, ..., u\}$ with $I_j(\varphi) \neq 0$, so that we may define

$$\overline{I}(\varphi) := \big[I_1(\varphi) : \cdots : I_u(\varphi)\big] \in \mathbf{h}_1.$$

This construction leads to the morphism

$$\begin{aligned} \chi_1\colon \mathcal{P}^\delta_{X/M/L/\varrho} &\longrightarrow \mathbf{h}_1 \\ [E, \varphi] &\longmapsto \overline{I}(\varphi). \end{aligned}$$

which we call the *Hitchin map* (see Page 24 for more details). Since the moduli space $\mathcal{P}^\delta_{X/M/L/\varrho}$ is projective, the Hitchin map is automatically projective.

Remark 2.4 *The above construction works also for negative a and the value $a = 0$. In the latter case, the line bundle M does not enter the definition of the Hitchin space, and we do not have to fix it, i.e., it suffices to fix just the degree of the participating vector bundles. The equivalence relation has to be formulated as for the example of the adjoint representation discussed in the next section.*

2.2 Example: Coherent Higgs systems

In the special case $s = 1$, the moduli space $\mathcal{P}^{\delta}_{X/M/L/\varrho}$ solves the classification problem for pairs (E, Γ) which consist of a vector bundle E and a one-dimensional subspace $\Gamma \subset H^0(X, E_\varrho \otimes L)$, i.e., for certain coherent systems. As pointed out in Remark 2.4, we may apply the results also in the case $a = 0$, and we do not need to fix the determinant in that case. In particular, we may apply them to the adjoint representation of $\mathrm{GL}_r(\mathbb{C})$ on $\mathrm{Mat}_r(\mathbb{C})$. Since this is the starting point of the present work, let us discuss this example in some detail.

The classification problem

We consider pairs (E, φ) that consist of a vector bundle E on X and a non-zero twisted endomorphism $\varphi\colon E \longrightarrow E \otimes L$. We say that (E_1, φ_1) is *isomorphic* to (E_2, φ_2), if there exist an isomorphism $\psi\colon E_1 \longrightarrow E_2$ and a non-zero complex number $\lambda \in \mathbb{C}^\star$ with

$$\varphi_2 = \lambda \cdot \big((\psi \otimes \mathrm{id}_L) \circ \varphi_1 \circ \psi^{-1}\big).$$

The set of isomorphism classes of such pairs clearly agrees with the set of isomorphism classes of coherent Higgs systems (E, Γ) with $\dim_{\mathbb{C}}(\Gamma) = 1$ presented in the introduction.

Asymptotic semistability

We let $\mathcal{H}^{\delta}_X(r, d)$ be the projective moduli space of δ-semistable pairs (E, φ) with $\mathrm{rk}(E) = r$ and $\deg(E) = d$. As in Section 2.1, we let $0 =: c_0 < c_1 < \cdots < c_t < c_{t+1} := \infty$ be the critical values for the concept of δ-semistability for pairs (E, φ) with $\deg(E) = d$. The two extremal cases have special properties.

Lemma 2.5 *Suppose that $\delta \in (c_0, c_1) \cap \mathbb{Q}$. Then, (E, φ) is δ-(semi)stable if and only if the following two conditions are satisfied.*

a) *The vector bundle E is semistable.*

b) *For every weighted filtration $(E_\bullet, \alpha_\bullet)$ of E with $\mu(E_i) = d/r$, $i = 1, ..., a$, one has $\mu(E_\bullet, \alpha_\bullet; \varphi)(\geq)0$.*

Remark 2.6 *Note that the lemma shows that (E, φ) is stable for $\delta \in (c_0, c_1) \cap \mathbb{Q}$, if E is a stable vector bundle. This shows that*

$$P(T^\star_{\mathcal{M}^{\mathrm{s}}_X(r,d)}) \subset \mathcal{H}^{\delta}_X(r, d).$$

Here, $\mathcal{M}^{\mathrm{s}}_X(r, d)$ is the moduli space of stable vector bundles of rank r and degree d on X, and $T^\star_{\mathcal{M}^{\mathrm{s}}_X(r,d)}$ is its cotangent bundle.

Lemma 2.7 *Suppose that $\delta \in (c_t, \infty) \cap \mathbb{Q}$. Then, (E, φ) is δ-(semi)stable if and only if the following two conditions are satisfied.*

a) *The twisted endomorphism φ is not nilpotent, i.e., $(\varphi \otimes \mathrm{id}_{L^{\otimes(r-1)}}) \circ \cdots \circ \varphi\colon E \longrightarrow E \otimes L^{\otimes r} \neq 0$.*

b) *For every subbundle $0 \subsetneq F \subsetneq E$ with $\varphi(F) \subset F \otimes L$, one has $\mu(F)(\geq)0$.*

Remark 2.8 *i) Part b) implies that (E, φ) is a semistable Higgs bundle. For $\delta \in (c_t, \infty)$, the moduli space $\mathcal{H}^\delta_X(r, d)$ is the divisor that compactifies Hitchin's moduli space $\mathfrak{H}_X(r, d)$ in the compactification constructed in [28]. It can be interpreted as a $\mathbb{C}^\star$-quotient of the latter, i.e.,*

$$\mathcal{H}^\delta_X(r, d) = \mathfrak{H}_X(r, d)/\!\!/\mathbb{C}^\star.$$

ii) Define $\mathbf{h} := \bigoplus_{i=1}^{r} H^0(X, L^{\otimes i})$ and let $\mathbb{C}^\star$ act on $H^0(X, L^{\otimes i})$ with weight i, $i = 1, ..., r$, Here, we may define $\mathbf{h}_1 := \mathbf{h}/\!\!/\mathbb{C}^\star$ as the resulting weighted projective space. Condition b) grants that there is a Hitchin map

$$\chi_1\colon \mathcal{H}^\delta_X(r, d) \longrightarrow \mathbf{h}_1.$$

It is automatically projective, because $\mathcal{H}^\delta_X(r, d)$ is projective.

The rank two case

A problem for applications might be that the Hitchin map χ_1 is not defined on the whole projectivized cotangent bundle $P(T^\star_{\mathcal{M}^{\mathrm{s}}_X(r,d)})$.

Example 2.9 *Let X be a curve of genus $g \geq 2$. We look at extensions*

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\iota} E \xrightarrow{\pi} \omega_X \longrightarrow 0.$$

These extensions are parameterized by $\mathrm{Ext}^1(\omega_X, \mathcal{O}_X) \cong H^1(X, \omega_X^\vee) \cong H^0(X, \omega_X^{\otimes 2})^\vee$. This is a vector space of dimension $3g - 3$. Proposition 2.4 in [17] and [16], Lemma 1, imply that there exist extensions for which E is stable. Pick an extension for which E is stable and define

$$\varphi := (\iota \otimes \mathrm{id}_{\omega_X}) \circ \pi\colon E \longrightarrow E \otimes \omega_X.$$

Then, (E, φ) defines a point in the projectivized cotangent bundle, but φ is clearly nilpotent, so that χ_1 is not defined at (E, φ).

There is the one parameter subgroup

$$\begin{array}{rcl} \lambda\colon \mathbb{C}^\star & \longrightarrow & \mathrm{SL}_2(\mathbb{C}) \\ z & \longmapsto & \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}. \end{array}$$

Note

$$\forall m = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \forall z \in \mathbb{C}^\star: \quad \lambda(z)\cdot m \cdot \lambda(z)^{-1} = \begin{pmatrix} a & z^{-2}\cdot b \\ z^2 \cdot c & d \end{pmatrix}.$$

This shows that, for a vector bundle E of rank two, a non-zero homomorphism $\varphi\colon E \longrightarrow E \otimes L$, and a sub line bundle $F \subset E$, one has

$$\mu(0 \subsetneq F \subsetneq E, (1); \varphi) = \begin{cases} -2, & \text{if } F \subset \mathrm{Ker}(\varphi) \\ 0, & \text{if } 0 \neq \varphi(F) \subset F \otimes L \\ 2, & \text{if } \varphi(F) \not\subset F \otimes L \end{cases}. \tag{2.9.1}$$

Example 2.10 *We return to the setting of Example 2.9. A sub line bundle $F \subset E$ distinct from $\mathcal{O}_X$ will not be invariant under φ. So, it follows from* (2.9.1) *that (E, φ) will be δ-(semi)stable if and only if the condition of δ-(semi)stability is satisfied for the sub line bundle $\mathcal{O}_X$ of E. This shows that (E, φ) is δ-stable for $\delta < g - 1$, properly $(g-1)$-semistable, and not δ-semistable for $\delta > g - 1$.*

2.3 Decorated pairs of vector bundles

We fix the same data as at the beginning of Section 2.1. A *holomorphic ϱ-triple* is a triple (F, E, φ) which consists of a vector bundle F of rank s on X, a vector bundle E of rank r on X, and a non-zero homomorphism $\varphi\colon F \longrightarrow E_\varrho \otimes L$.

Remark 2.11 *We may write F as $F' \otimes L$ for some other vector bundle F'. Then, φ corresponds to a homomorphism $\varphi'\colon F' \longrightarrow E_\varrho$. So, one might assume without loss of generality that $L = \mathcal{O}_X$.*

Semistability

The representation $\widetilde{\varrho}\colon \mathrm{GL}_s(\mathbb{C}) \times \mathrm{GL}_r(\mathbb{C}) \longrightarrow \mathrm{GL}(\mathbb{H})$ is bihomogeneous of bidegree $(-1, a)$, i.e., for all $z, w \in \mathbb{C}^\star$, $\widetilde{\varrho}(z \cdot E_s, w \cdot E_r) = z^{-1} \cdot w^a \cdot \mathrm{id}_{\mathbb{H}}$. Therefore, the results of Section 3 in [6] may be applied.

Let (F, E, φ) be a holomorphic ϱ-triple. For weighted filtrations $(F_\bullet, \beta_\bullet)$ and $(E_\bullet, \alpha_\bullet)$ of F and E, respectively, Formula (20) in [6] may be used to define

$$\mu\big((F_\bullet, \beta_\bullet), (E_\bullet, \alpha_\bullet); \varphi\big).$$

Fix a positive rational number δ. We say that a holomorphic ϱ-triple is δ-*(semi)stable*, if the inequality

$$M(F_\bullet, \beta_\bullet) + M(E_\bullet, \alpha_\bullet) + \delta \cdot \mu\big((F_\bullet, \beta_\bullet), (E_\bullet, \alpha_\bullet); \varphi\big)(\geq)0$$

is satisfied, for every pair $((F_\bullet, \beta_\bullet), (E_\bullet, \alpha_\bullet))$ of weighted filtrations of F and E.

For the following, we fix line bundles N and M on X. As before, there are critical values $0 =: c_0 < c_1 < \cdots < c_t < c_{t+1} := \infty$, such that the notions of δ-semistability and δ-stability for holomorphic ϱ-triples (F, E, φ) with $\det(F) \cong N$ and $\det(E) \cong M$ remain constant within each of the intervals $(c_i, c_{i+1}) \cap \mathbb{Q}$, $i = 0, ..., t$. We say that (F, E, φ) is *asymptotically (semi)stable*, if it is δ-(semi)stable with respect to a stability parameter $\delta \in (c_t, \infty) \cap \mathbb{Q}$.

Lemma 2.12 *A holomorphic ϱ-triple (F, E, φ) is asymptotically (semi)stable if and only if it satisfies the following two conditions.*

a) $\mu((F_\bullet, \beta_\bullet), (E_\bullet, \alpha_\bullet); \varphi) \geq 0$*, for all pairs* $((F_\bullet, \beta_\bullet), (E_\bullet, \alpha_\bullet))$ *of weighted filtrations,*

b) $M(F_\bullet, \beta_\bullet) + M(E_\bullet, \alpha_\bullet)(\geq)0$*, for all pairs* $((F_\bullet, \beta_\bullet), (E_\bullet, \alpha_\bullet))$ *of weighted filtrations with* $\mu((F_\bullet, \beta_\bullet), (E_\bullet, \alpha_\bullet); \varphi) = 0$*.*

Remark 2.13 *We use the notation from Remark 2.2. In addition, we define* $\mathbb{F}$ *as the restriction of* F *to* η*. So, the restriction of* φ *to* η *yields a point* $\varphi_\eta \in \mathrm{Hom}(\mathbb{F}, \mathbb{V})$*. The group* $\mathrm{SL}_s(\mathbb{K}) \times \mathrm{SL}_r(\mathbb{K})$ *acts on that vector space. The point* φ_η *will be semistable with respect to that group action if and only if Condition a) in Lemma 2.12 is satisfied. In that case, we call* φ generically semistable.

For $\delta \in \mathbb{Q}_{>0}$, we let $\mathcal{T}^\delta_{X/N/M/L/\varrho}$ be the moduli space of δ-semistable holomorphic ϱ-triples (F, E, φ) in which $\det(F) \cong N$ and $\det(E) \cong M$. For $\delta \in (c_t, \infty) \cap \mathbb{Q}$, there will be again a Hitchin map. Set $\mathbb{S} := \mathrm{SL}_s(\mathbb{C}) \times \mathrm{SL}_r(\mathbb{C})$. This group acts on $\mathbb{H} = \mathrm{Hom}(\mathbb{C}^s, V)$. This time, we will be interested in the invariant ring

$$\mathbb{C}[\mathbb{H}]^{\mathbb{S}} = \bigoplus_{d \geq 0} \mathrm{Sym}^d(\mathbb{H}^\vee)^{\mathbb{S}}.$$

Fix $k > 0$ and define

$$\widetilde{\mathbf{sym}}^\star_k := \bigoplus_{d \geq 0} \widetilde{\mathbf{sym}}^d_k, \quad \widetilde{\mathbf{sym}}^d_k := \mathrm{Sym}^{k \cdot d}(\mathbb{H}^\vee)^{\mathbb{S}}, \quad d \geq 0.$$

As before,

$$P(\mathbb{H})/\!\!/\mathbb{S} \cong \mathrm{Proj}(\widetilde{\mathbf{sym}}^\star_k), \quad k > 0.$$

We choose $k > 0$ in such a way that $\widetilde{\mathbf{sym}}_k^\star$ is generated by $\widetilde{\mathbf{sym}}_k^1$. In particular,

$$P(\mathbb{H})/\!\!/\mathbb{S} \hookrightarrow \mathbb{P}(\widetilde{\mathbf{sym}}_k^1).$$

Write $k = s \cdot n$ and $a \cdot k = r \cdot m$. The *Hitchin space* is now

$$\mathbf{h}_2 := \mathbb{P}\big(\widetilde{\mathbf{sym}}_k^1 \otimes H^0(X, N^{\otimes -n} \otimes M^{\otimes m} \otimes L^{\otimes k})\big). \tag{2.13.1}$$

For $\delta \in (c_t, \infty) \cap \mathbb{Q}$, we will have a *Hitchin map*

$$\chi_2 \colon \mathcal{T}^{\delta}_{X/N/M/L/\varrho} \longrightarrow \mathbf{h}_2.$$

In the next section, we will discuss examples of GIT problems which will help to understand Hitchin spaces and maps (see Section 5).

3 The underlying problem from geometric invariant theory

The understanding of the notion of semistability for coherent systems and the Hitchin map rests on the analysis of the underlying model given by the representation of the reductive affine algebraic group $\mathrm{SL}_s(\mathbb{C}) \times \mathrm{SL}_r(\mathbb{C})$ on the vector space $\mathrm{Hom}(\mathbb{C}^s, V)$ or rather its projectivization. As an illustration, we look at two specific examples.

Remark 3.1 *i) We may view the GIT problem just introduced as the problem of classifying coherent systems when the base variety is a point rather than a smooth projective curve X.*

ii) The GIT quotient can be formed in two steps, e.g., by first dividing by the action of the group $\mathrm{SL}_s(\mathbb{C})$ and then by the action of the group $\mathrm{SL}_r(\mathbb{C})$. This means that we may alternatively study the action of $\mathrm{SL}_r(\mathbb{C})$ on the Graßmannian $\mathfrak{G}$ of s-dimensional subspaces of V. We will see in Section 4.2 how these two viewpoints lead to different approaches for constructing moduli spaces for coherent systems on a smooth projective curve X which seem to work for different ranges of the stability parameter.

3.1 Symmetric tensors

We choose $a = s = r = 2$ and look at the subspace $\mathrm{Sym}^2(\mathbb{C}^2) \subset (\mathbb{C}^2)^{\otimes 2}$. Let us set $Q := \mathbb{C}^2$. Denote by (e_0, e_1) the standard basis of Q and by (x_0, x_1) the dual basis of $Q^\vee$. The space $\mathrm{Sym}^2(Q^\vee)$ is the space of quadratic forms on Q. We write a quadratic form $q \in \mathrm{Sym}^2(Q^\vee)$ as

$$q = q(x_0, x_1) = a \cdot x_0^2 + 2b \cdot x_0 x_1 + c \cdot x_1^2.$$

We may represent q also by the symmetric (2×2)-matrix

$$m_q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Let

$$\begin{aligned} \omega\colon\ \mathrm{SL}_2(\mathbb{C}) &\longrightarrow \mathrm{SL}_2(\mathbb{C}) \\ g &\longmapsto (g^{-1})^t \end{aligned}$$

be the outer automorphism. If $\tau\colon\ \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}(\mathrm{Sym}^2(Q^\vee))$ is the second symmetric power of the dual of the standard representation, the representation $\kappa := \tau\circ\omega$ is isomorphic to the second symmetric power of the standard representation of $\mathrm{SL}_2(\mathbb{C})$ on Q. So, we may write the action associated with the representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathrm{Sym}^2(Q)$ as

$$\begin{aligned} \mathrm{SL}_2(\mathbb{C})\times\mathrm{Sym}^2(Q^\vee) &\longrightarrow \mathrm{Sym}^2(Q^\vee) \\ (g,q) &\longmapsto \bigl(v\longmapsto q(g^t\cdot v)\bigr) \\ (g,m_q) &\longmapsto g\cdot m_q\cdot g^t. \end{aligned}$$

Next, note that there is the non-degenerate anti-symmetric pairing

$$\begin{array}{ccc} Q\times Q & \longrightarrow & \mathbb{C} \\ (v,w) & \longmapsto & \det(v|w). \end{array}$$

It is also invariant under the action of $\mathrm{SL}_2(\mathbb{C})$. So, it induces an isomorphism $Q\cong Q^\vee$ of representations of $\mathrm{SL}_2(\mathbb{C})$. We write $\mathbb{H} := \mathrm{Hom}(Q^\vee,\mathrm{Sym}^2(Q^\vee))$ and $\mathbb{S} := \mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})$. By our conventions, the $\mathbb{S}$-action A on $\mathbb{H}$ has the form

$$\begin{aligned} &\left(\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, g\right), (m_{q_1}, m_{q_2})\right) \longmapsto \qquad (3.1.1) \\ &\longmapsto \bigl(g\cdot(\alpha\cdot m_{q_1}+\gamma\cdot m_{q_2})\cdot g^t, g\cdot(\beta\cdot m_{q_1}+\delta\cdot m_{q_2})\cdot g^t\bigr). \end{aligned}$$

As recalled in Remark 3.1, ii), we may first form the quotient by the action of the second factor. For this, we need to determine the invariant ring

$$\mathbb{C}[\mathbb{H}]^{\mathrm{SL}_2(\mathbb{C})}.$$

It appears from time to time in the literature, but we haven't found a recent reference where it is explicitly determined. So, let us do it, here. We look at a point $(q_1,q_2)\in\mathbb{H}$ with

$$(m_{q_1}, m_{q_2}) = \left(\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}\right).$$

The first invariant is the determinant of m_{q_1}, i.e., we set

$$\Delta_1(q_1, q_2) := a_1 \cdot a_3 - a_2^2.$$

Likewise, the second invariant is given by the determinant of m_{q_2}, that is,

$$\Delta_2(q_1, q_2) := b_1 \cdot b_3 - b_2^2.$$

The third invariant is the so-called *codiscriminant*. It is defined by the formula

$$\Gamma(q_1, q_2) := \frac{1}{2} \cdot a_1 \cdot b_3 + \frac{1}{2} \cdot a_3 \cdot b_1 - a_2 \cdot b_2.$$

Remark 3.2 *i) Suppose that $I \in \mathbb{C}[\mathbb{H}]^{\mathrm{SL}_2(\mathbb{C})}$ is an invariant. Denoting by E_2 the identity matrix, we define, for $h \in \mathrm{SL}_2(\mathbb{C})$, the map*

$$\begin{aligned} h^\star(I)\colon \mathbb{H} &\longrightarrow \mathbb{H} \\ (q_1, q_2) &\longmapsto I\Big(A\big((h, E_2), (q_1, q_2)\big)\Big). \end{aligned}$$

Since the two actions of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{H}$ commute with each other, it follows that $h^\star(I)$ is also an element of $\mathbb{C}[\mathbb{H}]^{\mathrm{SL}_2(\mathbb{C})}$. Now, let

$$h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

Then, one computes

$$\begin{aligned} h^\star(\Delta_1) &= \alpha^2 \cdot \Delta_1 + \gamma^2 \cdot \Delta_2 + 2(\alpha\gamma) \cdot \Gamma, \\ h^\star(\Delta_2) &= \beta^2 \cdot \Delta_1 + \delta^2 \cdot \Delta_2 + 2(\beta\delta) \cdot \Gamma, \\ h^\star(\Gamma) &= (\alpha\beta) \cdot \Delta_1 + (\gamma\delta) \cdot \Delta_2 + (\alpha\delta + \beta\gamma) \cdot \Gamma. \end{aligned} \tag{3.2.1}$$

We can choose h, such that $\alpha\gamma \neq 0$. Since Δ_1, Δ_2, and $h^\star(\Delta_1)$ are invariants, the first formula shows that Γ is an invariant, too.

ii) The above computations are a special case of a procedure for computing the invariant rings for direct sums of representations (see [36]).

iii) We now explain what it means that all three invariants Δ_1, Δ_2, and Γ vanish at a point $(q_1, q_2) \in \mathbb{H}$. As is well known, Δ_i vanishes if and only if m_{q_i} has rank at most one, $i = 1, 2$. Recall that a (2×2)-matrix m defines the linear map

$$\begin{aligned} D_m\colon Q &\longrightarrow Q^\vee \\ v &\longmapsto \big(w \longmapsto v^t \cdot m \cdot w\big). \end{aligned}$$

The radical $\mathrm{Rad}(m)$ of m is the kernel of D_m. We infer that $\Delta_1(q_1, q_2) = \Delta_2(q_1, q_2) = \Gamma(q_1, q_2) = 0$ is equivalent to the following:

a) $\mathrm{rk}(m_{q_i}) \leq 1$, $i = 1, 2$, and,

b) if $q_1 \neq 0$ and $q_2 \neq 0$, then $\mathrm{Rad}(m_{q_1}) = \mathrm{Rad}(m_{q_2})$.

Proposition 3.3 *The invariants Δ_1, Δ_2, and Γ are algebraically independent and generate the invariant ring, i.e.,*

$$\mathbb{C}[\mathbb{H}]^{\mathrm{SL}_2(\mathbb{C})} = \mathbb{C}[\Delta_1, \Delta_2, \Gamma].$$

Proof. We will use Hilbert's "algorithm" for computing the invariant ring (see [11], [37], Section 4.6). We leave it to the reader to check that these invariants are algebraically independent. This will also be a consequence of the following discussion.

Abbreviate $R := \mathbb{C}[\Delta_1, \Delta_2, \Gamma]$ and $S := \mathbb{C}[\mathbb{H}]^{\mathrm{SL}_2(\mathbb{C})}$. The first thing to show is that Δ_1, Δ_2, and Γ cut out the nullforms. Below, we will determine the nullforms with respect to the action of the second factor of $\mathbb{S}$ with the Hilbert–Mumford criterion. It turns out that the nullforms are exactly those pairs satisfying the conditions stated in Remark 3.2, iii), i.e., exactly those pairs in which Δ_1, Δ_2, and Γ vanish. Then, according to [37], Theorem 4.6.1, $R \subset S$ is a finite ring extension. Since R is a normal ring, it suffices to show that the field extension $Q(R) \subset Q(S)$ has degree one. This amounts to showing that, for the morphism

$$\pi\colon \mathrm{Spec}(S) \longrightarrow \mathrm{Spec}(R),$$

there is a non-empty open subset $U \subset \mathrm{Spec}(S)$, such that $\pi^{-1}(U) \longrightarrow U$ is bijective. We look at pairs (q_1, q_2), such that $\Delta_1(q_1, q_2) \neq 0$, $\Delta_2(q_1, q_2) \neq 0$, and q_1 and q_2 have no common linear factor.

Since $\mathrm{SL}_2(\mathbb{C})$ acts triply transitively on $\mathbb{P}^1$, we may assume that there exist complex numbers $\lambda, \mu \in \mathbb{C}^\star$, and $a \in \mathbb{C} \setminus \{0, 1\}$ with

$$q_1 = \lambda \cdot x_0 x_1 \quad \text{and} \quad q_2 = \mu \cdot (x_0 - x_1) \cdot (x_0 - a \cdot x_1).$$

The matrix

$$g := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

transforms q_1 into $-\lambda \cdot x_0 x_1$ (and q_2 into $-a \cdot \mu \cdot (x_0 - x_1) \cdot (x_0 - (1/a) \cdot x_1)$). So, we may assume

$$\lambda \in \{\, z = u + v \cdot i \,|\, (v > 0) \vee (v = 0 \wedge u > 0) \,\}.$$

Under this assumption $\Delta_1(q_1, q_2) = -\lambda^2$ allows to reconstruct λ. Next,

$$\Delta_2(q_1, q_2) = -\frac{1}{4} \cdot \mu^2 \cdot (a - 1)^2, \quad \Gamma(q_1, q_2) = \lambda \cdot \mu \cdot (a + 1).$$

Since we know λ, we may determine the ratio

$$\left(\frac{a-1}{a+1}\right)^2.$$

Now, suppose that b is another complex number with

$$\left(\frac{a-1}{a+1}\right)^2 = \left(\frac{b-1}{b+1}\right)^2.$$

If

$$\frac{a-1}{a+1} = \frac{b-1}{b+1},$$

then $a = b$. Moreover, if $\mu \cdot (a+1) = \nu \cdot (b+1)$ and $a = b$, then also $\mu = \nu$. If, on the other hand,

$$\frac{a-1}{a+1} = \frac{1-b}{b+1},$$

then $b = 1/a$. The equality $\mu \cdot (a+1) = \nu \cdot (b+1)$ then yields $\nu = a \cdot \mu$. Pick a complex number c with $c^2 = a$. Then, the matrix

$$g := \begin{pmatrix} c & 0 \\ 0 & \frac{1}{c} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

leaves $q_1 = \lambda \cdot x_0 x_1$ invariant and transforms $q_2 = \mu \cdot (x_0 - x_1) \cdot (x_0 - a \cdot x_1)$ into $a \cdot \mu \cdot (x_0 - x_1) \cdot (x_0 - (1/a) \cdot x_1)$. So, under the assumptions on q_1 and q_2 made above, we may reconstruct the $\mathrm{SL}_2(\mathbb{C})$-orbit of (q_1, q_2) from $\Delta_1(q_1, q_2)$, $\Delta_2(q_1, q_2)$, and $\Gamma(q_1, q_2)$. ■

Proposition 3.4 *Set* $\Theta := \Delta_1 \cdot \Delta_2 - \Gamma^2$. *Then,*

$$\mathbb{C}[\mathbb{H}]^{\mathbb{S}} = \mathbb{C}[\Theta].$$

Proof. The formulas in (3.2.1) show that the action of the first factor of $\mathbb{S}$ on the vector space $\langle \Delta_1, \Delta_2, \Gamma \rangle$ is isomorphic to the representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathrm{Sym}^2(\mathbb{C}^2)$. The implies the assertion. ■

Remark 3.5 *As the first (or second) formula in* (3.2.1) *shows, the zeros of* Θ *correspond to pairs* $(q_1, q_2) \in \mathbb{H}$ *for which* $P(\langle m_{q_1}, m_{q_2} \rangle)$ *has exactly one point that corresponds to a matrix of rank one.*

Let us determine the canonical nullforms with the Hilbert–Mumford criterion. We define, for $\mu, \nu \geq 0$ and $\mu > 0$ or $\nu > 0$, the one parameter subgroup

$$\begin{aligned} \lambda_{\mu,\nu} \colon \mathbb{C}^\star &\longrightarrow \mathbb{S} \\ z &\longmapsto \left(\begin{pmatrix} z^{-\mu} & 0 \\ 0 & z^{\mu} \end{pmatrix}, \begin{pmatrix} z^{-\nu} & 0 \\ 0 & z^{\nu} \end{pmatrix} \right). \end{aligned}$$

Let us record the weights of the different entries of a pair of matrices associated with an element of $\mathbb{H}$:

$$\left(\begin{pmatrix} -\mu-\nu^2 & -\mu \\ -\mu & -\mu+\nu^2 \end{pmatrix}, \begin{pmatrix} \mu-\nu^2 & \mu \\ \mu & \mu+\nu^2 \end{pmatrix}\right).$$

Remark 3.6 *i) We are looking for non-zero pairs of* (2×2)*-matrices in* $\mathbb{H}$ *for which all weights are negative. If the second matrix is zero, this happens for* $\nu = 0$.

ii) If $\mu = 0$*, then all weights will be negative if and only if the only non-zero entry in both* m_{q_1} *and* m_{q_2} *is the top left one. One readily checks that* (q_1, q_2) *is a nullform with respect to the action of the second factor of* $\mathbb{S}$ *if and only if the conditions stated in Remark 3.2, iii), are satisfied.*

iii) Let us assume that both $q_1 \neq 0$ *and* $q_2 \neq 0$ *and that both* $\mu \neq 0$ *and* $\nu \neq 0$*. Then, in* m_{q_2}*, only the top left entry is non-zero and* $\mu < \nu^2$*. This forces the bottom right entry of* m_{q_1} *to be zero. In other words,* $\mathrm{Rad}(m_{q_2})$ *is an isotropic subspace for* m_{q_1}.

iv) If m_{q_1} *and* m_{q_2} *do not have a common isotropic subspace, then* (q_1, q_2) *is stable with respect to the action of the second factor of* $\mathbb{S}$.

Proposition 3.7 *A pair* $(q_1, q_2) \in \mathbb{H}$ *is semistable with respect to the action of* $\mathbb{S}$ *if and only if* a) $\langle\, q_1, q_2 \,\rangle$ *is a two-dimensional space and* b) m_{q_1} *and* m_{q_2} *have no common isotropic subspace.*

Proof. If (q_1, q_2) satisfies Condition a) and b), then it follows readily from the previous discussions that (q_1, q_2) is semistable. Now, suppose that (q_1, q_2) fails to meet Condition b). Since $(\alpha, \beta) \longmapsto \det(\alpha\cdot m_{q_1} + \beta\cdot m_{q_2})$ has to admit a zero, the vector space $\langle\, m_{q_1}, m_{q_2} \,\rangle$ has a non-zero element of rank one. By applying the action of the first factor of $\mathbb{S}$, we may assume without loss of generality that m_{q_2} has rank one. Then, the previous computations immediately show that (q_1, q_2) is a nullform. ■

The vector space $\mathbb{H}$ has dimension six, the group $\mathbb{S}$ has dimension six, as well, but the categorical quotient $\mathbb{H}/\!\!/\mathbb{S}$ has dimension one and not zero. This implies that there are no points in $\mathbb{H}$ which are stable with respect to the action of the group $\mathbb{S}$. Still, we have the following interesting observation.

Proposition 3.8 *Every point* $(q_1, q_2) \in \mathbb{H}$ *which is semistable with respect to the action of* $\mathbb{S}$ *is also polystable with respect to the action of* $\mathbb{S}$*. In particular, the quotient*

$$\mathbb{H} \longrightarrow \mathbb{H}/\!\!/\mathbb{S}$$

is a geometric quotient

Proof. A point $(q_1, q_2) \in \mathbb{H}$ is semistable (polystable) with respect to the action of $\mathbb{S}$ if and only if it is semistable (polystable) with respect to the action of the second factor $\mathbb{S}$ and the image of (q_1, q_2) in $\mathbb{H}/\!\!/ \mathrm{SL}_2(\mathbb{C})$ is semistable (polystable) with respect to the action of the first factor of $\mathbb{S}$ ([23], Proposition 1.3.1 and 1.3.2). Now, for the action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{H}/\!\!/ \mathrm{SL}_2(\mathbb{C}) \cong \mathrm{Sym}^2(Q^\vee)$, every point which is semistable is also polystable, and the latter happens, according to Proposition 3.7, if and only if $\langle\, m_{q_1}, m_{q_2}\,\rangle$ is a two-dimensional space and m_{q_1} and m_{q_2} have no common isotropic subspace. It follows from Remark 3.6, iv), that (q_1, q_2) is stable and, thus, polystable with respect to the action of the second factor of $\mathbb{S}$. ■

Remark 3.9 *Suppose that (q_1, q_2) is semistable in the above sense. We also need to know when the maximal weight is zero. If μ were zero, the lower right entry of both m_{q_1} and m_{q_2} would have to be zero. This is ruled out by semistability. So, the only possibility is that $\mu = \nu^2$, and, in the matrix m_{q_2}, only the top left entry is non-zero (and the lower right entry of the first matrix is non-zero).*

Example 3.10 *a) The point $(q_1, q_2) \in \mathbb{H}$ corresponding to the matrices*

$$m_{q_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad m_{q_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is semistable with respect to the action of the second factor of $\mathbb{S}$ but not with respect to the action of $\mathbb{S}$. In fact, m_{q_2} is - up to a scalar multiple - the only non-zero matrix in $\langle\, m_{q_1}, m_{q_2}\,\rangle$ of rank one.

b) The $\mathbb{S}$ orbit of a point $(q_1, q_2) \in \mathbb{H}$ which is semistable with respect to the action of $\mathbb{S}$ contains an element (r_1, r_2) with

$$m_{r_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad m_{r_2} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}.$$

Here,

$$a^2 = -4 \cdot \Theta(r_1, r_2) = -4 \cdot \Theta(q_1, q_2).$$

By requiring $\mathrm{Im}(a) > 0$ or $\mathrm{Im}(a) = 0$ and $a > 0$, we may make a unique.

3.2 Homomorphisms

For classical coherent systems, the model in geometric invariant theory would be the action of $\mathrm{GL}_s(\mathbb{C}) \times \mathrm{GL}_r(\mathbb{C})$ on the vector space $\mathrm{Hom}(\mathbb{C}^s, \mathbb{C}^r)$ and the quest for invariants with respect to the action of the group $\mathrm{SL}_s(\mathbb{C}) \times \mathrm{SL}_r(\mathbb{C})$. This problem is, unfortunately, rather uninteresting. Invariants exist only in the case $r = s$ and are given as polynomials in the determinant, in this case. So, let us first look at the action of $\mathrm{GL}_r(\mathbb{C})$ on the vector space $\mathrm{Hom}(\mathbb{C}^s, \mathbb{C}^r)$. The invariants for the action of the special linear group $\mathrm{SL}_r(\mathbb{C})$ are given by the first fundamental theorem of invariant theory for the special linear group.

Theorem 3.11 *For $s < r$, one has $\mathbb{C}[\mathrm{Hom}(\mathbb{C}^s, \mathbb{C}^r)]^{\mathrm{SL}_r(\mathbb{C})} = \mathbb{C}$, and, for $s \geq r$, the invariant ring is generated by the $(r \times r)$-minors.*

Proof. [5], Theorem 2.1. ■

Let us look at the special case $s = r + 1$. Then, $\mathbb{C}[\mathrm{Hom}(\mathbb{C}^s, \mathbb{C}^r)]^{\mathrm{SL}_r(\mathbb{C})}$ is the symmetric algebra of the vector space $\bigwedge^r \mathbb{C}^{r+1}$. Now, the wedge product

$$\mathbb{C}^{r+1} \otimes \bigwedge^r \mathbb{C}^{r+1} \longrightarrow \bigwedge^{r+1} \mathbb{C}^{r+1}$$

shows that, as a representation of $\mathrm{SL}_{r+1}(\mathbb{C})$, $\bigwedge^r \mathbb{C}^{r+1}$ is isomorphic to $(\mathbb{C}^{r+1})^\vee$.

In the construction of the Hitchin space $\mathbf{h}_3$ (4.6.2), we need to study the action of $\mathrm{SL}_{r+1}(\mathbb{C})$ on $((\mathbb{C}^{r+1})^\vee)^{\oplus h}$, for some $h > 0$. So, we may use Theorem 3.11 again, in order to do this.

Remark 3.12 *In connection with Hitchin spaces (see* (2.3.1) *and* (4.6.2)*), we need to study invariants in $S^{\oplus h}$ for an $\mathrm{SL}_s(\mathbb{C})$-module S and and $h > 0$. There are generalizations of the first fundamental theorem of invariant theory to this situation (see [36]). The module S itself is obtained from an invariant ring $\mathbb{C}[T]^{\mathrm{SL}_r(\mathbb{C})}$ by picking a system of generators and homogenizing. So, in order to find nice examples, one could start with situations where $\mathbb{C}[T]^{\mathrm{SL}_r(\mathbb{C})}$ is already a polynomial algebra. We refer to [25], §8, for an introduction to this topic. Interesting recent applications of such GIT problems are given in [2]. Some of the representations occurring there are of interest in the setting of coherent systems.*

4 Moduli spaces

We fix a homogeneous representation $\varrho\colon\ \mathrm{GL}_r(\mathbb{C}) \longrightarrow \mathrm{GL}(V)$, a positive integer s, line bundles L, M on X, and a stability parameter $\delta \in \mathbb{Q}_{>0}$. The aim is the definition of an appropriate notion of δ-semistability for coherent ϱ-systems and the sketch of a construction of moduli spaces. Particular care will be spent on the construction of the Hitchin map. An alternative approach has already been worked out in [34].

4.1 Review of the construction of moduli spaces of pairs

In this part, we will discuss some elements of the construction of moduli spaces of δ-semistable ϱ-pairs which are useful or even necessary for the understanding of the corresponding construction for moduli spaces of coherent systems. The definitions and constructions for holomorphic ϱ-triples are completely analogous.

The moduli functors

Let S be a scheme of finite type over $\mathbb{C}$. A *family of ϱ-pairs parameterized by S* is a tuple (E_S, L_S, φ_S) which consists of

- a vector bundle E_S of rank r on $S \times X$, such that there exists a line bundle M_S on S with $\det(E_S) \cong \pi_S^\star(M_S) \otimes \pi_X^\star(M)$,
- a line bundle L_S on S,
- and a homomorphism $\varphi_S \colon \mathcal{O}_{S\times X}^{\oplus s} \longrightarrow E_{S,\varrho} \otimes \pi_S^\star(L_S) \otimes \pi_X^\star(L)$.

We say that two families $(E_S^1, L_S^1, \varphi_S^1)$ and $(E_S^2, L_S^2, \varphi_S^2)$ of ϱ-pairs parameterized by S are *isomorphic*, if there exist isomorphisms

$$\psi_S \colon E_S^1 \longrightarrow E_S^2 \quad \text{and} \quad \lambda_S \colon L_S^1 \longrightarrow L_S^2,$$

such that

$$\varphi_S^2 = (\psi_{S,\varrho} \otimes \pi_S^\star(\lambda_S) \otimes \mathrm{id}_{\pi_X^\star(L)}) \circ \varphi_S^1.$$

Given a scheme S of finite type over $\mathbb{C}$, a family (E_S, L_S, φ_S) of ϱ-pairs parameterized by S, and a closed point $s \in S$, the ϱ-pair $(E_{S,s}, \varphi_{S,s})$ is obtained by restricting (E_S, L_S, φ_S) to $\{s\} \times X$. We say that (E_S, L_S, φ_S) is *δ-(semi)stable*, if $(E_{S,s}, \varphi_{S,s})$ is δ-(semi)stable, for every closed point $s \in S$. (This might involve defining δ-(semi)stability in the case that the residue field of s is not algebraically closed.) In this way, we obtain the moduli functors $P^{\delta\text{-(s)s}}_{X/M/L/\varrho}$ which assign to a scheme S of finite type over $\mathbb{C}$ the set of isomorphy classes of δ-(semi)stable families of ϱ-pairs parameterized by S. There is an obvious notion of the pullback of a family of ϱ-pairs via a morphism $f \colon T \longrightarrow S$ which enables us to define the map $P^{\delta\text{-(s)s}}_{X/M/L/\varrho}(f) \colon P^{\delta\text{-(s)s}}_{X/M/L/\varrho}(S) \longrightarrow P^{\delta\text{-(s)s}}_{X/M/L/\varrho}(T)$.

The parameter space and the group action

Fix $n \in \mathbb{N}$, set $p := d + r \cdot (n + 1 - g)$, $d := \deg(M)$, and choose a complex vector space W of dimension p. There exist a quasi-projective scheme

$$\mathfrak{P} := \mathfrak{P}_{X/M/L/p/\varrho},$$

line bundles $L_{\mathfrak{P}}$, $M_{\mathfrak{P}}$ on $\mathfrak{P}$, as well as families

$$q_{\mathfrak{P}}\colon W \otimes \pi_X^\star\big(\mathcal{O}_X(-n)\big) \longrightarrow E_{\mathfrak{P}}$$

and

$$\varphi_{\mathfrak{P}}\colon \mathcal{O}_{\mathfrak{P}\times X}^{\oplus s} \longrightarrow E_{\mathfrak{P},\varrho} \otimes \pi_{\mathfrak{P}}^\star(L_{\mathfrak{P}}) \otimes \pi_X^\star(L)$$

on $\mathfrak{P} \times X$, such that

- $E_{\mathfrak{P}}$ is a vector bundle of rank r on $\mathfrak{P} \times X$ with $\det(E_{\mathfrak{P}}) \cong \pi_{\mathfrak{P}}^\star(M_{\mathfrak{P}}) \otimes \pi_X^\star(M)$,
- $q_{\mathfrak{P}}$ is a surjective homomorphism,
- $\pi_{\mathfrak{P}\star}(q_{\mathfrak{P}} \otimes \mathrm{id}_{\pi_X^\star(\mathcal{O}_X(n))})\colon W \otimes \mathcal{O}_{\mathfrak{P}} \longrightarrow \pi_{\mathfrak{P}\star}(E_{\mathfrak{P}} \otimes \pi_X^\star(\mathcal{O}_X(n))$ is an isomorphism.

We will write points of $\mathfrak{P}$ in the form (q, φ). The (set-theoretic) map

$$\begin{aligned} A\colon\ &\mathrm{GL}_s(\mathbb{C}) \times \mathrm{GL}(W) \times \mathfrak{P} \longrightarrow \mathfrak{P} \\ &(g, h, q, \varphi) \longmapsto \big(q \circ (g^{-1} \otimes \mathrm{id}_{\pi_X^\star(\mathcal{O}_X(n))}), \varphi \circ (h^{-1} \otimes \mathrm{id}_{\mathcal{O}_{\mathfrak{P}\times X}})\big) \end{aligned} \tag{4.0.1}$$

underlies a scheme theoretic group action. We let

$$B\colon\ \mathrm{GL}(W) \times \mathfrak{P} \longrightarrow \mathfrak{P}$$

be the restriction of A to $\mathrm{GL}(W) \times \mathfrak{P}$.

If we choose n large enough, there exist $\mathrm{GL}(W)$-invariant open subsets $\mathfrak{P}^{\delta\text{-ss}}$ and $\mathfrak{P}^{\delta\text{-s}}$ of $\mathfrak{P}$, such that,

- for every point $s = (q\colon W \otimes \mathcal{O}_X(-n) \longrightarrow E, \varphi) \in \mathfrak{P}^{\delta\text{-(s)s}}$, the ϱ-pair (E, φ) is δ-(semi)stable,
- for every scheme S, every δ-(semi)stable family (E_S, L_S, φ_S) of ϱ-pairs parameterized by S, and every point $s \in S$, there exist an open neighborhood $s \in U \subset S$ and a morphism $\kappa_U\colon U \longrightarrow \mathfrak{P}^{\delta\text{-(s)s}}$, such that the restriction of (E_S, L_S, φ_S) to $U \times X$ is isomorphic to the pullback of the family $(E_{\mathfrak{P}}, L_{\mathfrak{P}}, \varphi_{\mathfrak{P}})$ via $\kappa_U \times \mathrm{id}_X$,
- for every scheme S, and every pair $\kappa_S^i\colon S \longrightarrow \mathfrak{P}$, $i = 1, 2$, of morphisms, such that the pullback $(E_S^1, L_S^1, \varphi_S^1)$ of $(E_{\mathfrak{P}}, L_{\mathfrak{P}}, \varphi_{\mathfrak{P}})$ via $\kappa_S^1 \times \mathrm{id}_X$ is isomorphic to the pullback $(E_S^2, L_S^2, \varphi_S^2)$ of $(E_{\mathfrak{P}}, L_{\mathfrak{P}}, \varphi_{\mathfrak{P}})$ via $\kappa_S^2 \times \mathrm{id}_X$, there exists a morphism $g_S\colon S \longrightarrow \mathrm{GL}(W)$, such that $\kappa_S^2 = B \circ (g_S \times \kappa_S^1)$.

These properties imply that the (coarse) moduli spaces for the functors $P^{\delta\text{-ss}}_{X/M/L/\varrho}$ and $P^{\delta\text{-s}}_{X/M/L/\varrho}$ are given by the categorical quotients

$$\mathcal{P}^{\delta}_{X/M/L/\varrho} := \mathfrak{P}^{\delta\text{-ss}} /\!\!/ \mathrm{GL}(W) \quad \text{and} \quad \mathcal{P}^{\delta\text{-s}}_{X/M/L/\varrho} := \mathfrak{P}^{\delta\text{-s}} /\!\!/ \mathrm{GL}(W).$$

Remark 4.1 *Since the center* $\mathbb{C}^\star \cdot \mathrm{id}_W \subset \mathrm{GL}(W)$ *acts trivially on* $\mathfrak{P}^{\delta\text{-(s)s}}$*, it suffices to prove the existence of the categorical quotients* $\mathfrak{P}^{\delta\text{-ss}} /\!\!/ \mathrm{SL}(W)$ *and* $\mathfrak{P}^{\delta\text{-s}} /\!\!/ \mathrm{SL}(W)$.

Construction of the Hitchin map

Now, assume that $\delta > c_t$, so that the notions of δ-semistability and asymptotic semistability are characterized by Lemma 2.1. For every invariant $I \in \mathbf{sym}^1_k$, we use the construction described after (2.3.1) to define a global section

$$I(\varphi_S)\colon \mathcal{O}_{\mathfrak{P}\times X} \longrightarrow \pi^\star_{\mathfrak{P}}(M^{\otimes l}_{\mathfrak{P}} \otimes L^{\otimes k}_{\mathfrak{P}}) \otimes \pi^\star_X(M^{\otimes l} \otimes L^{\otimes k}).$$

(Of course, we need to fix a suitable number k as described before.) We push this forward to $\mathfrak{P}$ and get

$$\mathcal{O}_{\mathfrak{P}} \longrightarrow H^0(X, M^{\otimes l} \otimes L^{\otimes k}) \otimes M^{\otimes l}_{\mathfrak{P}} \otimes L^{\otimes k}_{\mathfrak{P}}.$$

Choosing a basis $I_1, ..., I_u$ for $\mathbf{sym}^1_k$, the associated sections may be used to define a morphism

$$\widetilde{\chi}_{\mathfrak{P}}\colon \mathfrak{P}^{\delta\text{-ss}} \longrightarrow \mathbf{h}_1$$

with

$$\widetilde{\chi}^\star_{\mathfrak{P}}\big(\mathcal{O}_{\mathbf{h}_1}(1)\big) \cong M^{\otimes l}_{\mathfrak{P}} \otimes L^{\otimes k}_{\mathfrak{P}}.$$

This map is invariant under the action of $\mathrm{GL}(W)$. For this reason, it descends to a morphism

$$\chi_1\colon \mathcal{P}^{\delta}_{X/M/L/\varrho} \longrightarrow \mathbf{h}_1.$$

This is the Hitchin map.

4.2 The moduli problem for coherent systems

The moduli problem for coherent systems has already been discussed in our previous paper [34]. The parameter space for coherent ϱ-systems on which the GIT construction was performed in that source is itself a GIT quotient of the parameter space $\mathfrak{P}$ for ϱ-pairs from Section 4.1. This suggests that a construction of the moduli space for coherent ϱ-systems which is based on GIT on the parameter space $\mathfrak{P}$ should be possible. We will now try to explain this alternative construction.

Semistability

Let Γ be a finite dimensional complex vector space. A *weighted flag* in Γ is a pair $(\Gamma_\bullet, \alpha_\bullet)$ which consists of a (not necessarily complete) flag

$$0 \subsetneq \Gamma_1 \subsetneq \cdots \subsetneq \Gamma_b \subsetneq \Gamma$$

in Γ and a vector $\beta_\bullet = (\beta_1, ..., \beta_b)$ of positive rational numbers.

Remark 4.2 *A weighted flag $(\Gamma_\bullet, \beta_\bullet)$ determines a weighted filtration $(F_\bullet, \beta_\bullet)$ of $F := \Gamma \otimes \mathcal{O}_X$ with $F_i := \Gamma_i \otimes \mathcal{O}_X$, $i = 1, ..., b$. Clearly, not every weighted filtration of F comes from a weighted flag of Γ.*

Now, let (E, Γ) be a coherent ϱ-system. We define

$$\varphi_\Gamma \colon F = \Gamma \otimes \mathcal{O}_X \longrightarrow H^0(X, E_\varrho \otimes L) \otimes \mathcal{O}_X \xrightarrow{\text{ev}} E_\varrho \otimes L.$$

For a coherent ϱ-system (E, Γ), a weighted flag $(\Gamma_\bullet, \beta_\bullet)$ in Γ, and a weighted filtration $(E_\bullet, \alpha_\bullet)$ of E, we let $(F_\bullet, \beta_\bullet)$ be the associated weighted filtration of $F = \Gamma \otimes \mathcal{O}_X$ and set, using the definitions in Section 2.3,

$$\mu\big((\Gamma_\bullet, \beta_\bullet), (E_\bullet, \alpha_\bullet); \Gamma\big) := \mu\big((F_\bullet, \beta_\bullet), (E_\bullet, \alpha_\bullet); \varphi_\Gamma\big).$$

Let $\delta \in \mathbb{Q}_{>0}$ be a stability parameter. We call a coherent ϱ-system (E, Γ) δ-*(semi)stable*, if the inequality

$$M(E_\bullet, \alpha_\bullet) + \delta \cdot \mu\big((\Gamma_\bullet, \beta_\bullet), (E_\bullet, \alpha_\bullet); \Gamma\big) (\geq) 0$$

holds true, for every pair $((\Gamma_\bullet, \beta_\bullet), (E_\bullet, \alpha_\bullet))$, consisting of a weighted flag $(\Gamma_\bullet, \beta_\bullet)$ in Γ and a weighted filtration $(E_\bullet, \alpha_\bullet)$ of E.

Remark 4.3 *We may choose $\Gamma_\bullet = 0 \subsetneq \Gamma$ and $\beta_\bullet = ()$. In this case, the identity*

$$\mu\big((F_\bullet, \beta_\bullet), (E_\bullet, \alpha_\bullet); \Gamma\big) = \mu\big((E_\bullet, \alpha_\bullet); \varphi_\Gamma\big)$$

is satisfied, for every weighted filtration $(E_\bullet, \alpha_\bullet)$ of E. So, given a stability parameter $\delta \in \mathbb{Q}_{>0}$ and a δ-semistable coherent ϱ-system (E, Γ), it follows that the ϱ-pair (E, φ_Γ) is δ-semistable, as well.

Now, fix the rank r, the degree d, and the dimension s. The main theorem in [31] or Theorem 2.3.4.3 in [32] now implies that the family of isomorphy classes of vector bundles E of rank r and degree d for which there exists an s-dimensional subspace $\Gamma \subset H^0(X, E_\varrho \otimes L)$ and a stability parameter $\delta \in \mathbb{Q}_{>0}$, such that (E, Γ) is δ-semistable, is bounded.

In view of the strong boundedness result that we have just mentioned, we expect that the notion of δ-semistability for coherent ϱ-systems (E,Γ) with $\mathrm{rk}(E)=r$, $\deg(E)=d$, and $\dim_{\mathbb{C}}(\Gamma)=s$ stabilizes for large values of δ. We say that a coherent ϱ-system (E,Γ) is *asymptotically* (*semi*)*stable*, if

a) $\mu((F_\bullet,\beta_\bullet),(E_\bullet,\alpha_\bullet);\varphi_\Gamma)\geq 0$ holds, for every pair $((\Gamma_\bullet,\beta_\bullet),(E_\bullet,\alpha_\bullet))$, consisting of a weighted flag $(\Gamma_\bullet,\beta_\bullet)$ in Γ and a weighted filtration $(E_\bullet,\alpha_\bullet)$ of E,

b) for every pair $((\Gamma_\bullet,\beta_\bullet),(E_\bullet,\alpha_\bullet))$, consisting of a weighted flag $(\Gamma_\bullet,\beta_\bullet)$ in Γ and a weighted filtration $(E_\bullet,\alpha_\bullet)$ of E with $\mu((F_\bullet,\beta_\bullet),(E_\bullet,\alpha_\bullet);\varphi_\Gamma)=0$, one has $M(E_\bullet,\alpha_\bullet)(\geq)0$.

Problem 4.4 *Given r, d, and s as before, is it true that there is a positive rational number δ_∞, such that, for $\delta\geq\delta_\infty$, a coherent ϱ-system (E,Γ) with $\mathrm{rk}(E)=r$, $\deg(E)=d$, and $\dim_{\mathbb{C}}(\Gamma)=s$ is δ-(semi)stable if and only if it is asymptotically (semi)stable?*

In view of the above boundedness result, the proof should proceed along the lines of the proof of Proposition 2.3.6.5 in [32].

Remark 4.5 *As before, a coherent ϱ-system (E,Γ) defines a holomorphic ϱ-triple (F,E,φ_Γ), and, in the notation of Remark 2.2 and 2.13, a point $\varphi_{\Gamma,\eta}\in\mathrm{Hom}(\mathbb{K}^s,V_{\mathbb{K}})$. According to Remark 2.13, the first condition in the notion of asymptotic semistability for the holomorphic ϱ-triple (F,E,φ_Γ) requires that $\varphi_{\Gamma,\eta}$ is semistable with respect to the action of $\mathrm{SL}_s(\mathbb{K})\times\mathrm{SL}_r(\mathbb{K})$. In contrast, Condition a) in the definition of asymptotic semistability for coherent systems requires that $\varphi_{\Gamma,\eta}$ satisfies the Hilbert–Mumford criterion for pairs (λ',λ'') where $\lambda'\colon\mathbb{G}_m(\mathbb{C})\longrightarrow\mathrm{SL}_s(\mathbb{C})\subset\mathrm{SL}_s(\mathbb{K})$ and $\lambda''\colon\mathbb{G}_m(\mathbb{K})\longrightarrow\mathrm{SL}_s(\mathbb{K})$ are one parameter subgroups. In order to understand this notion with the help of geometric invariant theory, it will probably be helpful to restrict φ_Γ to a suitable open subset $U\subset X$ and choose appropriate trivializations in order to view $\varphi_{\Gamma|U}$ as a morphism $U\longrightarrow\mathrm{Hom}(\mathbb{C}^s,V)$ as in [30], Lemma 1.4.*

The moduli functors

Let S be a scheme of finite type over $\mathbb{C}$. A *family of coherent ϱ-systems parameterized by S* is a family (E_S,L_S,φ_S) of ϱ-pairs, such that

$$\pi_{S\star}(\varphi_S)\colon\Gamma\otimes\mathcal{O}_S\longrightarrow\pi_{S\star}\big(E_{S,\varrho}\otimes\pi_X^\star(L)\big)\otimes L_S$$

is universally injective, i.e., for every base change diagram

$$\begin{tikzcd} T\times X \arrow[r, "f\times \mathrm{id}_X"] \arrow[d, "\pi_T"'] & S\times X \arrow[d, "\pi_S"] \\ T \arrow[r, "f"] & S \end{tikzcd}, \tag{4.5.1}$$

the homomorphism

$$\pi_{T\star}\bigl((f\times \mathrm{id}_X)^\star(\varphi_S)\bigr)\colon \Gamma\otimes\mathcal{O}_T\longrightarrow \pi_{T\star}\bigl((f\times \mathrm{id}_X)^\star(E_{S,\varrho})\otimes\pi_X^\star(L)\bigr)\otimes f^\star(L_S)$$

is injective.

Remark 4.6 *Let us explain the notion of universal injectivity from a different viewpoint. First, we note that we may find an ample line bundle N on X, such that $\pi_{S\star}(E_{S,\varrho}\otimes\pi_X^\star(N))\otimes L_S$ is locally free and commutes with base change. We may also fix an injective homomorphism $L\longrightarrow N$. For a base change diagram as* (4.5.1)*, we get the commutative diagram ([8], Chapter III, Remark 9.3.1):*

$$\begin{tikzcd} \Gamma\otimes\mathcal{O}_T \arrow[r] \arrow[d] & f^\star\bigl(\pi_{S\star}\bigl(E_{S,\varrho}\otimes\pi_X^\star(L)\bigr)\otimes L_S\bigr) \arrow[r] \arrow[d] & f^\star\bigl(\pi_{S\star}\bigl(E_{S,\varrho}\otimes\pi_X^\star(N)\bigr)\otimes L_S\bigr) \arrow[d] \\ \Gamma\otimes\mathcal{O}_T \arrow[r] & \pi_{T\star}\bigl(f_X^\star(E_{S,\varrho})\otimes\pi_X^\star(L)\bigr)\otimes f^\star(L_S) \arrow[r] & \pi_{T\star}\bigl(f_X^\star(E_{S,\varrho})\otimes\pi_X^\star(N)\bigr)\otimes f^\star(L_S) \end{tikzcd}.$$

Here, we have set $f_X := f\times \mathrm{id}_X$. We point out the following:

- *Since the projection $T\times X\longrightarrow X$ is flat, the homomorphism $\pi_T^\star(L)\longrightarrow \pi_T^\star(N)$ is injective. The sheaf $E_{S,\varrho}$ is locally free, so that the homomorphism $f_X^\star(E_{S,\varrho})\otimes\pi_X^\star(L)\longrightarrow f_X^\star(E_{S,\varrho})\otimes\pi_X^\star(N)$ is injective. It follows that the second homomorphism in the bottom row of the above diagram is always injective. This means that the homomorphism $\pi_{S\star}(\varphi_S)$ is universally injective if and only if the corresponding homomorphism*

  $$\psi_S\colon \Gamma\otimes\mathcal{O}_S\longrightarrow \pi_{S\star}\bigl(E_{S,\varrho}\otimes\pi_X^\star(N)\bigr)\otimes L_S$$

  is universally injective.

- *Assume that S is reduced. Using the outer square in the above diagram and the fact that the leftmost and the rightmost vertical homomorphism are isomorphisms, it follows that ψ_S is universally injective if and only if $\Gamma\otimes\mathcal{O}_S$ is a subbundle of $\pi_{S\star}(E_{S,\varrho}\otimes\pi_X^\star(N))\otimes L_S$.*

- *Suppose that S is reduced and (E_S, L_S, φ_S) is an arbitrary family of ϱ-pairs paramterized by S. As before, let $\psi_S\colon \Gamma \otimes \mathcal{O}_S \longrightarrow \pi_{S\star}(E_{S,\varrho} \otimes \pi_X^\star(N)) \otimes L_S$ be the induced homomorphism. Then, there is the open subscheme U of points $s \in S$, for which $\psi_{S|\{s\}\times X}$ has maximal rank. The foregoing discussion shows that, for the family (E_U, L_U, φ_U) of ϱ-pairs that is obtained by restricting (E_S, L_S, φ_S) to $U \times X$, the homomorphism $\pi_{U\star}(\varphi_U)$ is universally injective.*

Two families $(E^1_S, L^1_S, \varphi^1_S)$ and $(E^2_S, L^2_S, \varphi^2_S)$ parameterized by the scheme S are *isomorphic*, if there exist an isomorphism $\psi_S\colon E^1_S \longrightarrow E^2_S$ and an isomorphism $\lambda_S \in \Gamma(S, \mathrm{GL}_s(\mathcal{O}_S))$, such that

$$\varphi^2_S = (\psi_{S,\varrho} \otimes \mathrm{id}_{\pi_X^\star(L)}) \circ \varphi^1_S \circ \lambda_S^{-1}.$$

Fix $\delta \in \mathbb{Q}_{>0}$. It is now clear how to define the moduli functors $C^{\delta\text{-(s)s}}_{X/M/L/\varrho}$ of δ-(semi)stable coherent ϱ-systems.

The construction of the moduli spaces

We start with the parameter scheme $\mathfrak{P}$ from Section 4.1. It is reduced. According to Remark 4.6, there is an open subscheme $\mathfrak{C} \subset \mathfrak{P}$ that is characterized by the property that, for the universal family $(E_{\mathfrak{C}}, L_{\mathfrak{C}}, \varphi_{\mathfrak{C}})$ that is obtained by restricting the universal family $(E_{\mathfrak{P}}, L_{\mathfrak{P}}, \varphi_{\mathfrak{P}})$ to $\mathfrak{C} \times X$, the associated homomorphism $\pi_{\mathfrak{C}\star}(\varphi_{\mathfrak{C}})$ is universally injective. The action defined in (4.0.1) induces an action

$$\mathrm{GL}_s(\mathbb{C}) \times \mathrm{GL}(W) \times \mathfrak{C} \longrightarrow \mathfrak{C}.$$

Using the techniques from the paper [6], it is now possible to carry out the program outlined for ϱ-pairs in Section 4.1 also for coherent ϱ-systems, i.e., we find the $(\mathrm{GL}_s(\mathbb{C}) \times \mathrm{GL}(W))$-invariant open subsets $\mathfrak{C}^{\delta\text{-ss}}$ and $\mathfrak{C}^{\delta\text{-s}}$ that parameterize the coherent ϱ-systems that are δ-semistable and δ-stable, and the categorical quotients

$$\mathcal{C}^{\delta}_{X/M/L/\varrho} := \mathfrak{C}^{\delta\text{-ss}} /\!\!/ \big(\mathrm{GL}_s(C) \times \mathrm{GL}(W)\big)$$

and

$$\mathcal{C}^{\delta\text{-s}}_{X/M/L/\varrho} := \mathfrak{C}^{\delta\text{-s}} /\!\!/ \big(\mathrm{GL}_s(\mathbb{C}) \times \mathrm{GL}(W)\big)$$

do exist and constitute the moduli spaces for the functors $C^{\delta\text{-ss}}$ and $C^{\delta\text{-s}}$, respectively.

The Hitchin map

According to [23], Proposition 1.3.1, the moduli space $\mathcal{C}^\delta_{X/M/L/\varrho}$ may be constructed as a $\mathrm{GL}_s(\mathbb{C})$-quotient of

$$\mathcal{P}^\delta_{X/M/L/\varrho} = \mathfrak{P}^{\delta\text{-ss}} /\!\!/ \mathrm{GL}(W).$$

By the same result, $\mathrm{GL}_s(\mathbb{C})$ acts on $\mathbf{sym}^1_k$ and, thus, on $\mathbf{h}_1$ (2.3.1). The Hitchin map

$$\chi_1 \colon \mathcal{P}^\delta_{X/M/L/\varrho} \longrightarrow \mathbf{h}_1 \tag{4.6.1}$$

is $\mathrm{GL}_s(\mathbb{C})$-equivariant.

Now, we define

$$\mathbf{h}_3 := \mathbf{h}_1 /\!\!/ \mathrm{GL}_s(\mathbb{C}). \tag{4.6.2}$$

Let us be a bit more precise. The center of $\mathrm{GL}_s(\mathbb{C})$ acts trivially on $\mathbf{h}_1$. So, as before,

$$\mathbf{h}_1 /\!\!/ \mathrm{GL}_s(\mathbb{C}) = \mathbf{h}_1 /\!\!/ \mathrm{SL}_s(\mathbb{C}).$$

The $\mathrm{GL}_s(\mathbb{C})$-action and, so, also the $\mathrm{SL}_s(\mathbb{C})$-action on $\mathbf{h}_1$ is naturally linearized in the line bundle $\mathcal{O}_{\mathbf{h}_1}(1)$. The above quotient is then to be understood as the quotient constructed by Mumford for the linearized $\mathrm{SL}_s(\mathbb{C})$-action. Note that, if the center of $\mathrm{GL}_s(\mathbb{C})$ acts with non-zero weight on

$$\boldsymbol{s}_1 := \mathbf{sym}^1_k \otimes H^0(X, M^{\otimes l} \otimes L^{\otimes k}),$$

then there are no points which are $\mathrm{GL}_s(\mathbb{C})$-semistable in the sense of Mumford. This is why we are working with the action of $\mathrm{SL}_s(\mathbb{C})$.

Remark 4.7 *As a representation of* $\mathrm{GL}_s(\mathbb{C})$*,* $\boldsymbol{s}_1$ *is isomorphic to the direct sum of* $\dim_{\mathbb{C}}(H^0(X, M^{\otimes l} \otimes L^{\otimes k}))$ *copies of* $\mathbf{sym}^1_k$*. So, if one knows* $\mathbf{sym}^1_k$ *as a* $\mathrm{GL}_s(\mathbb{C})$*-module, then one may use the techniques described, e.g., in [36] in order to determine the Hitchin space* $\mathbf{h}_3$*. Examples were presented in Section 3.*

We expect that (4.6.1) induces a Hitchin map on $\mathcal{C}^\delta_{X/M/L/\varrho}$ for suitable values of δ. So, we formulate the following.

Problem 4.8 *Let* $\mathbf{h}^{\mathrm{ss}}_1 \subset \mathbf{h}_1$ *be the* $\mathrm{SL}_s(\mathbb{C})$*-invariant open subset of points that are* $\mathrm{SL}_s(\mathbb{C})$*-semistable in the sense of Mumford, and let*

$$H \colon \mathfrak{P}^{\delta\text{-}ss} \longrightarrow \mathcal{P}_{X/M/L/\varrho} \xrightarrow{\mathbf{h}_1} \mathbf{h}_1$$

be the quotient morphism followed by the Hitchin map. Is it true that, for $\delta \gg 0$*,*

$$\mathfrak{C}^\delta \subset H^{-1}(\mathbf{h}^{\mathrm{ss}}_1)?$$

Even if the answer to the above problem were negative, it would make sense to study the Hitchin map and the Hitchin space. Indeed, in order to construct the moduli space $\mathcal{C}^\delta$ as a GIT quotient of $\mathfrak{P}^{\delta\text{-ss}}$, we have used a linearization of the group action in some ample line bundle $\mathfrak{L}_1$. Furthermore, the pullback $\mathfrak{L}_2$ of $\mathcal{O}_{\mathbf{h}_1}(1)$ via the equivariant morphism H is also linearized. Relative geometric invariant theory ([27], Theorem 2.1, [33], Proposition 1.2) shows that, for large values of $c \in \mathbb{N}$, the set $\mathfrak{C}^\circ$ of points that are semistable with respect to the linearization in $\mathfrak{L}_1 \otimes \mathfrak{L}_2^{\otimes c}$ is contained in $H^{-1}(\mathbf{h}_1^{\mathrm{ss}})$. The quotient $\mathcal{C}^\circ := \mathfrak{C}^\circ /\!\!/ (\mathrm{GL}_s(\mathbb{C}) \times \mathrm{GL}(W))$ will then be endowed with a projective Hitchin map

$$\chi_3 \colon \mathcal{C}^\circ \longrightarrow \mathbf{h}_3.$$

5 Examples

We will discuss some very basic examples which mainly serve to illustrate properties of the Hitchin map. They pick up the topics of Section 2.3. We will compare them with the corresponding examples of holomorphic triples.

5.1 Symmetric tensors

We let X be a smooth projective curve and L a line bundle on X. Consider a vector bundle E of rank two and a subspace $\Gamma \subset H^0(X, \mathrm{Sym}^2(E^\vee) \otimes L)$ of dimension two. The test objects are pairs, consisting of a one dimensional subspace $\Lambda \subset \Gamma$ together with a positive rational number ν and a sub line bundle $N \subset E$ together with a positive rational number μ, corresponding to the weighted flag $(0 \subsetneq \Lambda \subsetneq \Gamma, (\nu))$ and the weighted filtration $(0 \subsetneq N \subsetneq E, (\mu))$, respectively. We use the computations from Page 18f for the homomorphism $\varphi_\Gamma = (\beta_1, \beta_2) \colon \mathcal{O}_X^{\oplus 2} \longrightarrow \mathrm{Sym}^2(E^\vee) \otimes L$. For Condition a) in the definition of asymptotic (semi)stability, we use the discussion leading to Proposition 3.7.

Remark 5.1 *There is an important point to observe when drawing the conclusions from the computations. A homomorphism $\beta \colon \mathrm{Sym}^2(E) \longrightarrow L$ is given by a symmetric (2×2)-matrix*

$$m_\beta = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$$

with $s_1, s_2, s_3 \in H^0(X, L)$. Suppose that $s_1, s_2 \in H^0(X, L)$ are linearly independent sections and choose

$$m_{\beta_1} = \begin{pmatrix} s_1 & 0 \\ 0 & s_1 \end{pmatrix} \quad \textit{and} \quad m_{\beta_2} = \begin{pmatrix} 0 & s_2 \\ s_2 & 0 \end{pmatrix}.$$

For $\lambda_1, \lambda_2 \in \mathbb{C}$, not both zero, the discriminant of $\lambda_1 \cdot m_{\beta_1} + \lambda_2 \cdot m_{\beta_2}$ is

$$\lambda_1^2 \cdot s_1^2 - \lambda_2^2 \cdot s_2^2 \not\equiv 0.$$

Therefore, every element in $\Gamma := \langle \beta_1, \beta_2 \rangle$ has generic rank two, in contrast to the observation in the proof of Proposition 3.7.

The first property that we need is that β_1 and β_2 are linearly independent over $\mathbb{C}$. This is ensured by the fact that Γ is a two-dimensional subspace. The second condition requires that there is no sub line bundle $N \subset E$ which is isotropic for both β_1 and β_2, i.e., isotropic for all $\beta \in \Gamma$, and occurs as the radical of an element $\beta \in \Gamma \setminus \{0\}$. Finally, we point out that the one parameter subgroups with weight zero described in Remark 3.9 are associated with an element $q \in \langle q_1, q_2 \rangle \setminus \{0\}$ and the radical $\Lambda \subset \mathbb{C}^2$ of q. In addition, the case $\mu = 0$ may occur here and corresponds to a sub line bundle $N \subset E$ which is isotropic for all $\beta \in \Gamma$. Our discussion yields the following criterion.

Lemma 5.2 *The coherent system (E, Γ) is asymptotically (semi)stable if and only if*

a) *there is no sub line bundle $N \subset E$ which is isotropic for all $\beta \in \Gamma$ and agrees with the radical of an element $\beta \in \Gamma \setminus \{0\}$, and*

b) *for every sub line bundle $N \subset E$ which is either isotropic for all $\beta \in \Gamma$ or occurs as the radical of one element $\beta \in \Gamma \setminus \{0\}$, the inequality*

$$\deg(N)(\leq)\mu(E) = \frac{\deg(E)}{2}$$

is satisfied.

Example 5.3 *In the example presented in Remark 5.1, the radical of every element in Γ is trivial and there is no sub line bundle which is isotropic for all $\beta \in \Gamma$. This means that (E, Γ) is asymptotically stable.*

Remark 5.4 *i) The Hitchin space $\mathbf{h}_1$ for pairs $(E, \varphi\colon \mathcal{O}_X^{\oplus 2} \to \operatorname{Sym}^2(E^\vee) \otimes L)$ is*

$$P\big(H^0(X, (M^\vee)^{\otimes 4} \otimes L^{\otimes 2})^{\oplus 3}\big).$$

The three summands correspond to the invariants Δ_1, Δ_2, and Γ, respectively.
ii) Let (E, Γ) be a coherent system which fails to meet Condition a) in Lemma 5.2. Assume that the sub line bundle $N \subset E$ is the radical of β_1 and isotropic for β_2. As usual, let $\mathbb{K} = \mathbb{C}(X)$ be the function field of X. We choose a trivialization

$E_{|\{\eta\}} \cong \mathbb{K}^2$ of E at the generic point of X and apply a suitable transformation from $\mathrm{SL}_2(\mathbb{K})$, such that, up to a factor in $\mathbb{K}^\star$, we get

$$m_{\beta_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad m_{\beta_2} = \begin{pmatrix} 1 & \star \\ \star & 0 \end{pmatrix}.$$

If $\star$ is zero, then the pair $(E, \varphi_\Gamma \colon \mathcal{O}_X^{\oplus 2} \longrightarrow \mathrm{Sym}^2(E^\vee) \otimes L)$ is not generically semistable (see Remark 2.2), and the invariants Δ_1, Δ_2, and Γ all vanish at (E, φ_Γ), so that the Hitchin map χ_1 is not defined at (E, φ_Γ). If $\star$ is different from zero, then the Hitchin map χ_1 is defined at (E, φ_Γ) and lies in the summand corresponding to Δ_1. Using the arguments from Section 3, it is readily checked that these elements are nullforms for the action of $\mathrm{SL}_2(\mathbb{C})$. So, the Hitchin map χ_3 is not defined at (E, Γ).

When looking at holomorphic triples $\varphi \colon F \longrightarrow \mathrm{Sym}^2(E^\vee) \otimes L$, the results of the section on symmetric tensors in Section 3 give the notion of generic semistability. By Proposition 3.4, the invariant ring is generated by a single invariant of degree four. So, according to (2.13.1), the base for the Hitchin map for holomorphic triples (F, E, φ) with $\mathrm{rk}(F) = \mathrm{rk}(E) = 2$, $\det(E) \cong M$, and $\det(F) \cong \mathcal{O}_X$ is

$$\mathbf{h}_2 = P\big(H^0(X, (M^\vee)^{\otimes 8} \otimes L^{\otimes 4})\big).$$

In contrast, the Hitchin space for coherent systems (E, Γ) is the $\mathrm{SL}_2(\mathbb{C})$-quotient of

$$\mathbf{h}_1 = P\big(\mathrm{Sym}^2(\mathbb{C}^2) \otimes H^0(X, (M^\vee)^{\otimes 4} \otimes L^{\otimes 2})\big).$$

The following example illustrates the difference between the notions of asymptotic semistability for the two species of objects.

Example 5.5 *Let us choose $X = \mathbb{P}^1$, $L = \mathcal{O}_X(1)$, and $E = \mathcal{O}_X^{\oplus 2}$. We let (s_1, s_2) be the usual basis of $H^0(X, L)$. Pick*

$$m_{\beta_1} = \begin{pmatrix} s_1 & s_1 \\ s_1 & 0 \end{pmatrix} \quad \textit{and} \quad m_{\beta_2} = \begin{pmatrix} s_2 & s_2 \\ s_2 & 0 \end{pmatrix}.$$

The holomorphic triple given by these data is not generically semistable, but the coherent system (E, Γ) is asymptotically semistable. We compute

$$\Delta_1(\beta_1, \beta_2) = -s_1^2, \quad \Delta_2(\beta_1, \beta_2) = -s_2^2, \quad \textit{and} \quad \Gamma(\beta_1, \beta_2) = -s_1 \cdot s_2.$$

Using $(s_1^2, s_2^2, s_1 \cdot s_2)$ as a basis of $H^0(X, L^{\otimes 2})$ and identifying $(\Delta_1, \Delta_2, \Gamma)$ with the basis $(e_1^2, e_2^2, e_1 \cdot e_2)$ for $\mathrm{Sym}^2(\mathbb{C}^2)$, as in the proof of Proposition 3.4, we see that we have

$$\chi_1\big([E, \varphi_\Gamma]\big) = \big[(e_1^2, 0, 0), (0, e_2^2, 0), (0, 0, e_1 \cdot e_2)\big] \in \mathbf{h}_1.$$

Since $[e_1 \cdot e_2] \in P(\mathrm{Sym}^2(\mathbb{C}^2))$ *is semistable for the action of* $\mathrm{SL}_2(\mathbb{C})$*, the point* $\chi_1([E, \varphi_\Gamma])$ *is semistable for the action of* $\mathrm{SL}_2(\mathbb{C})$*, and the Hitchin map* χ_3 *is defined at* $[E, \Gamma]$.

5.2 Classical coherent systems

Let X be a smooth projective curve and L a line bundle on X. Here, we look at pairs (E, Γ) which consist of a holomorphic vector bundle E on X and a subspace $\Gamma \subset H^0(X, E \otimes L)$.

Remark 5.6 *Replacing* E *by* $E \otimes L$*, we may actually assume* $L = \mathcal{O}_X$*. The reader may check that the condition of* δ*-(semi)stability is not affected by this manipulation. In this way, we may comply with the usual conventions in the literature. We will not make use of this in the example below to keep the notation closer to the general setting.*

The condition of (semi)stability introduced by Raghavendra and Vishwanath [26] says that, for $\delta \in \mathbb{Q}_{>0}$, a coherent system (E, Γ) is δ-(semi)stable, if, for every subbundle $0 \subsetneq G \subsetneq E$ and every subspace $\Gamma' \subset \Gamma \cap H^0(X, G \otimes L)$, the inequality

$$\frac{\deg(G) + \delta \cdot \dim_{\mathbb{C}}(\Gamma')}{\mathrm{rk}(G)} (\leq) \frac{\deg(E) + \delta \cdot \dim_{\mathbb{C}}(\Gamma)}{\mathrm{rk}(E)}$$

is satisfied. We checked that the general notion of δ-(semi)stability for coherent ϱ-pairs that we defined in [34], Section 1.4, specializes to this one, for $\varrho = \mathrm{id}\colon \mathrm{GL}_r(\mathbb{C}) \longrightarrow \mathrm{GL}(\mathbb{C}^r)$ (see [34], Example 1.4.1). We also note that, in this special case, there is no restriction on the stability parameter in the construction from [34].

Problem 5.7 *Does the notion of* δ*-semistability for coherent* ϱ*-systems defined in Section 4.2 specialize to the above condition for* $\varrho = \mathrm{id}$*?*

This will probably follow with the help of the arguments used in [29], Section 3.

In the sequel, we will work relative to $X = \mathbb{P}^1$ and choose $L := \mathcal{O}_X(1)$. Pick a basis (s_1, s_2) for $H^0(X, L)$. We look at the surjection $\varphi\colon \mathcal{O}_X^{\oplus 3} \longrightarrow E$, $E := \mathcal{O}_X^{\oplus 2}$, given by the matrix

$$m := \begin{pmatrix} s_1 & 0 & s_1 + s_2 \\ 0 & s_2 & s_1 + s_2 \end{pmatrix}.$$

The three minors are $s_1 \cdot s_2$, $s_1^2 + s_1 \cdot s_2$, and $s_1 \cdot s_2 + s_2^2$. They form a basis for $H^0(X, L^{\otimes 2})$. (The fact that they don't have a common zero shows that φ

is surjective.) Define $\Gamma \subset H^0(X, E \otimes L)$ as the subspace that is spanned by the columns of the matrix m. We claim that (E, Γ) is δ-stable, for all $\delta \in \mathbb{Q}_{>0}$. This amounts to the fact that there is no sub line bundle $N \subset E$ with $\dim_{\mathbb{C}}(\Gamma_N) = 2$, $\Gamma_N := \Gamma \cap H^0(X, N \otimes L)$. For such a line bundle, we would find the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & \mathit{\Gamma}_N \otimes \mathcal{O}_X \arrow[r] \arrow[d] & \mathit{\Gamma} \otimes \mathcal{O}_X \arrow[r] \arrow[d] & (\mathit{\Gamma}/\mathit{\Gamma}_N) \otimes \mathcal{O}_X \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\ 0 \arrow[r] & N \otimes L \arrow[r] & E \otimes L \arrow[r] & (E/N) \otimes L \arrow[r] & 0 \end{tikzcd}.$$

The snake lemma implies that $(E/N) \otimes L$ is globally generated by a single section, i.e., $(E/N) \otimes L \cong \mathcal{O}_X$. The fact that $(E/N) \otimes L$ is a quotient of $E \otimes L$ implies that $(E/N) \otimes L \cong \mathcal{O}_X(k)$ with $k \geq 1$. This is a contradiction.

Remark 5.8 *This example and more general versions can be found in [4], Section 5.4, and [18], Example 6.6.*

The Hitchin space is the $\mathrm{SL}_3(\mathbb{C})$-quotient of

$$P\big(V^\vee \otimes H^0(X, L)\big) \cong P\big((V^\vee)^{\oplus 3}\big), \quad V := \mathbb{C}^3,$$

i.e.,

$$\mathbf{h}_3 \cong \star.$$

The Hitchin map χ_3 is defined at the isomorphy class of (E, Γ). According to [18], Example 6.6, the moduli space contains just $[E, \Gamma]$. So, the Hitchin map is, in fact, an isomorphism.

Remark 5.9 *i) If we replace $\mathcal{O}_X(1)$ by $\mathcal{O}_X(k)$ with $k \geq 2$, we get positive dimensional Hitchin spaces, by Theorem 3.11. Using an embedding $\mathcal{O}_X(1) \longrightarrow \mathcal{O}_X(k)$, we can extend the above example. We will still get a stable coherent system. In that case, $\Gamma \otimes \mathcal{O}_X \longrightarrow E \otimes \mathcal{O}_X(k)$ is only generically surjective.*

ii) Let $\mathbb{K} \cong \mathbb{C}(t)$ be the function field of X. There are no points in the projective space $P(\mathrm{Hom}(\mathbb{K}^3, \mathbb{K}^2))$ which are semistable with respect to the action of $\mathrm{SL}_3(\mathbb{K}) \times \mathrm{SL}_2(\mathbb{K})$. This means that for fixed degrees e and d and $\delta \gg 0$, there are no δ-semistable holomorphic triples $\varphi\colon F \longrightarrow E$ with $\mathrm{rk}(F) = 3$, $\deg(F) = e$, $\mathrm{rk}(E) = 2$, and $\deg(E) = 2$ (compare Lemma 2.12 and Remark 2.13).

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