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DISCRETE SAMPLING THEOREM TO SHANNON'S SAMPLING THEOREM USING THE HYPERREAL NUMBERS $^*\mathbb{R}$

DEL TEOREMA DEL MUESTREO DISCRETO A TEOREMA DEL MUESTREO DE SHANNON MEDIANTE LOS NÚMEROS HIPERREALES $^*\mathbb{R}$

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Abstract

Shannon's sampling theorem is one of the most important results of modern signal theory. It describes the reconstruction of any band-limited signal from a finite number of its samples. On the other hand, although less well known, there is the discrete sampling theorem, proved by Cooley while he was working on the development of an algorithm to speed up the calculations of the discrete Fourier transform. Cooley showed that a sampled signal can be resampled by selecting a smaller number of samples, which reduces computational cost. Then it is possible to reconstruct the original sampled signal using a reverse process. In principle, the two theorems are not related. However, in this paper we will show that in the context of Non-Standard Mathematical Analysis (NSA) and Hyperreal Numerical System $^*\mathbb{R}$, the two theorems are equivalent. The difference between them becomes a matter of scale. With the scale changes that the hyperreal number system allows, the discrete variables and functions become continuous, and Shannon's sampling theorem emerges from the discrete sampling theorem.

Keywords: Sampling theorem; subsampling; hyperreal number system; infinitesimal calculus model.

Resumen

El teorema del muestreo de Shannon es uno de los resultados más importantes de la moderna teoría de señales. Este describe la reconstrucción de toda señal de banda limitada desde un número finito de sus muestras. Por otra parte, aunque menos conocido, se tiene el teorema del muestreo discreto, demostrado por Cooley mientras trabajaba en la elaboración de un algoritmo para acelerar los cálculos de la transformada discreta de Fourier. Cooley demostró que una señal muestreada se puede volver a muestrearla mediante la selección de un número menor de muestras, lo cual reduce el costo computacional. Luego, es posible reconstruir la señal muestreada original mediante un proceso inverso. En principio, los dos teoremas no están relacionados. Sin embargo, en este artíclo demostraremos que, en el contexto del Análisis Matemático No Estándar (ANS) y el Sistema Numérico Hiperreal $^*\mathbb{R}$, los dos teoremas son equivalentes. La diferencia entre ellos se vuelve un asunto de escala. Con los cambios de escala que permite realizar el sistema numérico hiperreal, las variables y funciones discretas se vuelven continuas, y el teorema del muestreo de Shannon emerge del teorema del muestreo discreto.

Palabras clave: Teorema de Muestreo; Submuestreo; Sistema Numérico Hiperreal; Modelo de Cálculo Infinitesimal.

Mathematics Subject Classification: 94D02

1 Introduction

The Shannon's sampling theorem is one of the most important results obtained by classical communication theory. It has a significant impact on the communication of information [20]. The theorem allows us to reconstruct the original signal by interpolating the samples. The original sampling theorem was proposed to be applied to limited band signals exclusively. However, the theorem has evolved to be used in band-pass signals or multi-band signals [18].

The theorem in principle was developed from a practical engineering approach, nowadays theoretical elaborations have been constructed from the perspective of vector spaces, establishing the theorem within the framework of mathematical rigor [15, 16].

A couple of decades after the original formulation of the sampling theorem, while working on the development of an efficient computational algorithm for the calculation of the Discrete Fourier Transform, Cooley and others developed a version of the theorem that can be applied to sampled signals [2]. In essence, a sampled signal has a specified number of samples. Cooley and his partner's finding was that fewer samples can be selected to represent the sampled signal, and all of the original samples can be retrieved without loss of information.

In principle, the two theorems are unrelated. In fact, Cooley makes no mention of Shannon's theorem. However, in this paper we are going to prove that both theorems are the same, but with the variables involved on different scales. The main result of this paper is the proof of both theorems are actually one. For this, the mathematical expressions resulting from the theorems are supported in the hyperreal numerical system $^*\mathbb{R}$, of non-standard mathematical analysis (NSA) [19, 3, 4]. Basically, we start with the finite-scale discrete sampling theorem, for discrete functions and variables. Then, the theorem is placed on an infinitesimal scale, from which continuous variables and functions are derived, and Shannon's sampling theorem emerges heuristically.

The paper is organized as follows. Section 2 presents the classical Shannon's sampling theorem and dicrete sampling theorem, developed by Cooley et al. The result of the theorem is presented without proof, because it is widely known and can be found in the cited reference. In section 3 we present a synthesis of the relevant definitions and properties of the NSA, as well as the elements of a calculus model inspired by Leibniz. In section 4 the conversion of discrete signals in time and frequency into continuous signals is performed through the NSA, and its application in the conversion of the sampling theorem of signals sampled into the classical sampling theorem. Finally, the conclusions are presented.

2 Sampling theorems for continuous signals and discrete signals

This section summarizes Shannon's sampling theorem and the discrete sampling theorem. The proofs are omitted. For the proof of Shannon's theorem you can see [10, 8]. For the rigorous proof of the discrete theorem, you can see [2], and in [6] a more didactic version of it is developed.

2.1 Shannon's sampling theorem

In communication theory, the signal, as carrier entity of information, is modeled mathematically as a function that can take values in $\mathbb{C}$ supported by two domains, that is, the time domain and the frequency domain, which take values in $\mathbb{R}$. The Shannon's sampling theorem states that the complete information of a limited band signal in the continuous domain is determined by the values of a countable number of its samples.

Theorem 1 *If a function $f(t)$ contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced $1/2W$ seconds apart [20].*

Due to similar contributions from various authors, this result has been named by Jerri [11], WKS sampling theorem, in honor of Whittaker, Shannon and Kotel'nikov. Or Nyquist-Shannon theorem due to the contribution of Nyquist [17].

Considering that the frequency spectrum of a signal is determined by its Fourier transform, Brigham [1] formulates the Shannon theorem as follows.

Theorem 2 *Let $x(t)$ be a function such that its Fourier transform $X(f)$ exists and it satisfies the condition $X(f) = 0$ outside the interval $[-W, W]$, then $x(t)$ can be expressed as*

$$x(t) = \sum_{q=-\infty}^{\infty} x\left[\frac{q}{2W}\right] \frac{\sin \pi(2Wt - q)}{\pi(2Wt - q)}. \tag{1}$$

The values $x[q/2W]$ are called the samples of the function in $q/2W$. The value of the last frequency W is called the bandwidth. The function $x(t)$ for which its spectrum meets $X(f) = 0$ as long as $f \geq W$ is called the band-limited signal and the signal represented by $x(t)$ is said to be covered by its samples. The critical interval $[-W, W]$ in the frequency domain, outside which coverage is not guaranteed, is called the Nyquist interval.

2.2 Discrete domain sampling

Is it possible to represent a phenomenon of discrete domain in a discrete subdomain without loss of information? The answer is positive. It was answered by Cooley et al with the discrete sampling theorem [2].

A new question arises, is there a relationship between the discrete version and Shannon's original sampling theorem? We will prove that there is a model in NSA that allows the transition from the discrete domain to the continuous domain and links the two results. In other words, the sampling theorem in its discrete version is exactly the same as the continuous version, if the phenomenon is represented in a calculus model based on the hyperreal numerical system, ${}^*\mathbb{R}$. The original of our approach is that we are going to reformulate the sampling theorem in a non-standard calculus model, which will allow us to analyze the origin of the discrete domain of the Shannon's sampling theorem and its transition to the continuous domain. Our result is based on the reciprocal relationship between the time domain and the frequency domain, as well as the double relationship between the periodicity of the domains and the sampling of the signals.

The discrete time sampling theorem involves several concepts, so we consider it convenient to separate them, to facilitate their understanding. First, the construction of the discrete domains in which the signals are studied is carried out, and its connection through the variables time and frequency. We then proceed to show the discrete Fourier transform for the variables corresponding to the time and frequency domains previously constructed. Finally, we proceed to present a educational version of Cooley's discrete sampling theorem.

2.2.1 The discrete time-frequency domain

Consider the interval $[-T, T]$ given by the real number $0 < T$. Let N be a positive integers and $-N \leq n < N$. We will call the time domain the set

$$\left\{\frac{nT}{N}\right\} = \left\{-\frac{NT}{N}, \ldots, -\frac{T}{N}, 0, \frac{T}{N}, \ldots, \frac{(N-1)T}{N}\right\}. \tag{2}$$

This is a domain of $2N$ elements or samples, whose temporal variable is

$$t = \{t_n\} = \left\{\frac{nT}{N}\right\}. \tag{3}$$

The $2T$ quantity is the period or duration of the time interval. Similarly, as the quotient $1/2T$ is the number of times that $2T$ fits in unity, and this is precisely the definition of frequency, the discrete frequency domain of $2N$ elements or frequencies, where $-N \leq k < N$, is the set

$$\left\{\frac{k}{2T}\right\} = \left\{-\frac{-N}{2T}, \ldots, -\frac{1}{2T}, 0, \frac{1}{2T}, \ldots, \frac{(N-1)}{2T}\right\}. \tag{4}$$

The frequency variable f of the domain is

$$f = \{f_k\} = \left\{\frac{k}{2T}\right\}. \tag{5}$$

In summary, as long as we have $0 < T$, N positive integer and $-N \leq n, k < N$, we can construct the pair of discrete domains

$$\left\{\frac{nT}{N}\right\}, \quad \left\{\frac{k}{2T}\right\}. \tag{6}$$

The variables of both domains are related

$$t_n = \frac{nT}{N}, \quad f_k = \frac{k}{2T}, \quad t_n f_k = \frac{nk}{2N}. \tag{7}$$

In this last expression, the inverse relationship between the two variables is measured. For a detailed treatment of this construction of domains see [7].

2.2.2 Discrete Fourier transform

We will denote as $x[nT/N]$ any complex value function over the time domain $\{nT/N\}$. A new function is determined $X[k/2T]$ of complex values on the frequency domain $\{k/2T\}$, for the double relationship

$$X\left[\frac{k}{2T}\right] = \frac{1}{2N}\sum_{n=-N}^{N-1} x\left[\frac{nT}{N}\right] e^{-j2\pi\frac{nk}{2N}}, \tag{8}$$

$$x\left[\frac{nT}{N}\right] = \sum_{k=-N}^{N-1} X\left[\frac{k}{2T}\right] e^{j2\pi\frac{nk}{2N}}. \tag{9}$$

The first expression is called the discrete Fourier transform of $x[nT/N]$ and the second, inverse discrete Fourier transform of $X[k/2T]$, respectively. This last function is known as the frequency spectrum of that one, which is called the waveform. According to texts such as Brigham's, applying a well-known result of the complex finite geometric series and to Euler's formula $e^{(j2\pi N)} = 1$, it can be verified that $x[nT/N]$ is exactly the inverse of $X[k/2T]$, that is, if we replace one in the other, we get the identity.

2.2.3 A discrete version of the sampling theorem

The double relationship between discrete periodicity and sampling determines a surprising interpolation result, very similar to Shannon's theorem, which can be considered its discrete version. To develop this theorem, we must first establish the concept of bandwidth in the discrete domain. We will follow Cooley's original idea. Suppose that the function $x[nT/N]$ represents a signal that has a frequency spectrum that meets the condition $X[k/2T] = 0$, for $M < |k|$, where $M \leq N$. Since $M/2T$ is a frequency above which the spectrum $X[k/2T]$ is zero, we will call $M/2T$ the signal bandwidth. Henceforth, we will assume that the integer M meets the condition $N = MP$. So let's fix a function of which we only know that in the frequency domain it is limited band.

Because the frequency spectrum is restricted to its bandwidth, whatever the values of the waveform, they are necessarily expressed as

$$x\left[\frac{nT}{N}\right] = \sum_{k=-M}^{M} X\left[\frac{k}{2T}\right] e^{j2\pi\frac{nk}{2N}}. \tag{10}$$

Now we apply the relationships between sampling and periodicity, which we have previously found. As the frequency band of $X[k/2T]$ is limited to the range $-M \leq k < M$ we can think that such a frequency spectrum is periodic of period $2M$, which is copied P times in its frequency domain. This periodic spectrum corresponds to a function in the time domain sampled with $2M$ samples. Therefore, we can try to retrieve each of the $2N$ values of the original waveform from their temporal values in the $2M$ samples $x[qPT/N] = x[qT/M]$. To do this, we replace the equation (10) in the equation (11), and we get

$$x\left[\frac{nT}{N}\right] = \sum_{k=-M}^{M} \frac{1}{P}\frac{1}{2M} \sum_{q=-M}^{M-1} x\left[\frac{qT}{M}\right] e^{-j2\pi\frac{qk}{2M}} e^{j2\pi\frac{nk}{2N}}. \tag{11}$$

Since $N = MP$

$$-\frac{qk}{2M} + \frac{nk}{2N} = \frac{(n-qP)k}{2N}. \tag{12}$$

If we use the traditional notation of W for the bandwidth, and we make $W = M/2T$, we have that $T/M = 1/2W$. Reordering

$$x\left[\frac{nT}{N}\right] = \frac{1}{2N} \sum_{q=-M}^{M-1} x\left[\frac{q}{2W}\right] \sum_{k=M}^{M} e^{j2\pi\frac{(n-qP)k}{2N}}. \tag{13}$$

This last equation would be the discrete version of Shannon's sampling theorem. All of the $2N$ values of the original sampled function are reconstructed with a subset of $2M$ samples. Furthermore, it is highlighted that the interval between samples in the time domain is $T/M = 1/2W$, which is the well-known Nyquist interval of the sampling theorem in the continuous domain.

3 Hyperreal numerical system and infinitesimal calculus

The NSA provides the hyperreal numerical system. When the traditional elements of the calculus are supported in the hyperreal numerical system, then an infinitesimal calculus model is obtained. These elements are variables, differential, functions, derivatives and integrals. For a detailed treatment of it, see [3, 4, 6, 9, 12, 13, 19]. In this section, a brief overview of the hyperreal numerical system and the elements of an infinitesimal calculus model.

3.1 Hyperreal numerical system

The following development shows that every extension of the real field has infinitesimal and infinite numbers. In literature it is known that there is a number system called the field of real numbers, whose symbol is $\mathbb{R}$. Its elements are denoted with letters such as $p, q, r \in \mathbb{R}$. It is less known that there are extensions of $\mathbb{R}$, numerical systems that contain it, in a very special sense, because they are linearly ordered fields. This type of field is usually symbolized as ${}^*\mathbb{R}$ and its elements ${}^*p, {}^* q, {}^* r \in {}^*\mathbb{R}$. This indicates that $\mathbb{R} \subset {}^*\mathbb{R}$ is true, where the second also has a linear order. Furthermore, the valid formulas in one of them are valid in the other, according to the principle transfer presented by Robinson.

Transfer Principle. If for all $r \in \mathbb{R}$, $P(r)$ is true, where P is an internal statement or formula well formed of a first order language, then $P({}^*r)$ is true for all ${}^*r \in {}^*\mathbb{R}$.

That is, every function or relation in $\mathbb{R}$ is a function or relation in ${}^*\mathbb{R}$. Any formula that is satisfied in one field is satisfied in the other field. So, for example, the sum and product, the polynomials, sinusoids and exponential functions in $\mathbb{R}$ are also defined in ${}^*\mathbb{R}$. Formulas such as the triangular inequality $|p + q| \leq |p| + |q|$ are not only true for $p, q \in \mathbb{R}$ but also for ${}^*p, {}^* q \in {}^*\mathbb{R}$, thus, $|{}^*p +{}^* q| \leq |{}^*p| + |{}^*q|$.

Accepting that there is a linearly ordered field $\mathbb{R} \subset {}^*\mathbb{R}$ has profound arithmetic and logical consequences. Here it will be shown that such a field will have not only finite elements, but also infinites and infinitesimals. Since this is not the case for the real number system, this would mean that the field $\mathbb{R}$, which has been constructed to hold all kinds of quantities, integers, rational and irrational, is not as large as imagined, but simply, the largest field that does not have infinites and infinitesimals.

Definition 1 *The infinitesimal numbers are the numbers that satisfy the condition of being smaller in magnitude than any real number, that is,* $|\epsilon| < |r|$*. Where* ϵ *is an infinitesimal hyperreal and* r *is a real. On the other hand, given an infinitesimal* α*, write* $\alpha \approx 0$*, and say that* α *is infinitely close to zero. It is accepted that 0 is the only real number that is also an infinitesimal number. Infinitesimals are symbolized by lowercase Greek letters* α*,* β*,* δ*,* ϵ*, etc.*

Definition 2 *Infinite hyperreals, symbolized with capital letters, are the hyperreals that in absolute value are greater than any real number, that is, for all real* r *then* $|r| < |M|$*.* M *is infinite hyperreal. A particular case is constituted by the infinite integers, called hyper-integer.*

Definition 3 *Finite hyperreals numbers are the hyperreals between two consecutive real numbers, that is,* $u <^* r < v$*. Where* u *and* v *are reals, and* ${}^*r$ *is a finite hyperreal. Hyperreals are symbolized with lowercase letters like this,* ${}^*r, {}^* u, {}^* v$*, etc.*

The hyperreal numbers fulfill the properties presented below.

Property 1. The result of adding two infinitesimal numbers is another infinitesimal, that is, $\alpha + \beta = \gamma$, for α, β and γ infinitesimals.
Property 2. The result of multiplying two infinitesimal numbers is another infinitesimal, that is, $\alpha \cdot \beta = \gamma$, for α, β and γ infinitesimals.
Property 3. The result of multiplying a finite hyperreal number by an infinitesimal is an infinitesimal number, that is, ${}^*r \cdot \alpha = \gamma$, for α, γ infinitesimals and ${}^*r$ finite hyperreal.
Property 4. The infinite hyperreal numbers M are multiplicative inverses of the nonzero infinitesimals.
Property 5. The multiplicative inverse of a finite hyperreal number is another finite hyperreal number.
Property 6. The standard part of the sum of two hyperreal numbers is equal to the sum of the standard parts of each of the hyperreal numbers.
Property 7. The standard part of the product of two hyperreal numbers is equal to the product of the standard parts of each of the hyperreal numbers.

Finally, it must be added that $^*\mathbb{R}$ is a non-archimedean field, by not fulfilling the archimedean property of the field $\mathbb{R}$. This property consists of the fact that no matter how small r is, some integer multiple of r reaches and will exceed u, no matter how large u is. In $^*\mathbb{R}$, the number one could never reach an infinite hyperreal by a finite integer multiple.

This also occurs on an infinitesimal scale: since every finite integer multiple of an infinitesimal is infinitesimal, never a finite integer multiple of an infinitesimal could reach a real.

In conclusion, the existence of a fully ordered field $^*\mathbb{R}$ that satisfies $\mathbb{R} \subset^* \mathbb{R}$, implies, necessarily, the existence of quantities of an arithmetic nature different from that of the real ones: the infinitesimals, the finites, and the infinites.

3.2 A calculus model based on the hyperreal numerical system

Now we proceed to the presentation of some essential concepts of a calculus model based on the Hyperreal Numerical System, such as, variable, continuity, and integrals. This model is fully developed in [5, 6]. Basically it consists of taking the most important elements of the calculus model developed by Leibniz and placing it in the context of the $^*\mathbb{R}$ field, providing rigor required by formal mathematics.

3.2.1 Variables

Signals can be defined simply as electrical and mathematical representations of the physical variables that can be found in nature and in human-made processes. The definition of variable presented in this section has been developed in [6] and applied in [21].

Definition 4 *An arbitrary variable x is an ordered sequence of values or numerical data of the form:*

$$x = \{x_0, x_1, x_2, \ldots, x_N\}\,. \tag{14}$$

where N is an infinite positive integer, which is called the variable range. Sometimes it is convenient to indicate the terms of the variable starting them in negative subscripts, $x = \{x_{-N}, x_{-N+1}, ..., x_0, \ldots, x_{N-1}, x_N\}$.

Each of the x_n is an infinitesimal, finite, or infinite hyperreal, and corresponds to a term or data in position n, also called the index n, suggesting the idea of a variable that goes through all its positions, from the first to the last.

To facilitate the assimilation by the students, the representation of the variable will be done with the traditional notation of the sequences:

$$x = \{x_n\} \qquad 0 \leq n \leq N. \tag{15}$$

This notation indicates that the variable on the left is exactly the same as the sequence on the right, which has as a general term x_n, which could be given by a formula. Due to the Robinson transfer principle, each hyperreal-valued variable has an equivalent real-valued variable. To obtain the equivalent real-valued variable, the standard part of the hyperreal-valued variable must be taken, $st(x) = \big\{st(x_{n)}\big\}$.

A very simple example of a variable is to subdivide the hyperreal interval $(0, 1)$ into N parts, where N is an infinite integer. Each position is $x_n = n/N$, so the variable is $\{0, 1/N, 2/N, 3/, \ldots, N/N = 1\}$. It is clear that the variable can be range in infinitesimal increments. Let the infinitesimal $\alpha = 1/N$, for N infinite hyperinteger. Then the variable will be determined by the sequence $x = \{n\alpha \mid 0 \leq n \leq N\} = \{0, \alpha, 2\alpha, \ldots, (N-1)\alpha, 1\}$.

In general, the variable can range an arbitrary hyperreal interval (a, b), for $a, b \in {}^*\mathbb{R}$. Let the infinitesimal $\alpha = (b-a)/N$, for N infinite hyperinteger. The sequence will be $x = \{x_n \mid 0 \leq n \leq N\}$, whose general term is $x_n = n(b-a)/N + a$.

At this point, the differential of a variable can also be defined.

Definition 5 *The differential of a variable x, is the variable resulting from finding the difference between all the successive values of x, $\mathrm{d}x_n = x_{n+1} - x_n$. With $\mathrm{d}x = \{\mathrm{d}x_0, \mathrm{d}x_1, \ldots, \mathrm{d}x_N\} = \{\mathrm{d}x_n\}$.*

The last term of the new variable $\mathrm{d}x$ is made zero, so that the differential also has $N + 1$ terms. Various characteristics of the variable depend on this differential.

Definition 6 *If the differential of a variable is constant, the variable is said to be independent.*

Definition 7 *If the differential of a variable is infinitesimal, the variable is said to be continuous.*

If $\mathrm{d}x$ takes the same infinitesimal value in all its positions, $\mathrm{d}x = \{\alpha, \alpha, ..., \alpha\}$, then we have an independent and continuous variable, which is widely used in engineering applications. An example is variable time that appears in physics and engineering problems.

3.2.2 Continuity of variables

We then proceed to establish, from the infinitesimal perspective, when a variable is continuous or discrete. In standard calculus, a variable is considered continuous when there is no bounce or hole in the linearly ordered set of values it takes.

In the Infinitesimal Calculus, since the variables have been defined as sequences, there will always be a bounce in each one of the values, because this sequence is discrete, even if it is on an infinitesimal scale.

So, a variable is continuous if the difference between two successive values of the variable is an infinitesimal:

$$x_{n+1} - x_n \approx 0. \tag{16}$$

The following is a specific case of a variable, which will be useful in this paper. We consider a finite interval of the hyperreal line ${}^*\mathbb{R}$, of the form $[-L, L]$, where L is real finite hyperreal, and the finite integer N. Which results in a sequence $x = \{nL/N\}$, $-N \leq n \leq N$, which is a discrete variable with a finite number of elements. If N is now taken as an infinite integer, then:

$$x = \{x_n\} = \left\{\frac{nL}{N}\right\}. \tag{17}$$

Assuming the infinitesimal:

$$\mathrm{d}x = \frac{L}{N}. \tag{18}$$

It allows rewriting the variable of the form:

$$x = \{x_n\} = \{n\mathrm{d}x\}. \tag{19}$$

The difference of two successive values of the variable:

$$x_{n+1} - x_n = (n+1)\mathrm{d}x - n\mathrm{d}x = \mathrm{d}x. \tag{20}$$

is infinitesimal, indicating that the variable is continuous. If $y_i = f(x_i)$ is done, we have that $\mathrm{d}y = f(x + \mathrm{d}x) - f(x)$, very notation usual, since Leibniz, who founded the differential and integral calculus.

In summary, in the presented calculus model, a continuous variable takes values in an infinite sequence whose terms are infinitesimally separated. When the separation is not infinitesimal, the variable is said to be non-continuous or discontinuous.

3.2.3 Functions

Let $u = \{u_n\}$ and $v = \{v_n\}$, two variables in the sense handled in the infinitesimal calculus, for $0 \leq n \leq N$. The ordered pair (u, v) is called a function. In other words, a function is a relation between variables determined only by the fact of being presented in the order u, v. Considering the pair in that order, we will say that v is a function of u, and we write $v = f(u)$, where u is the argument or variable of the domain and v is the variable of the range.

The functional nature of the relationship between the two variables is that each term of the first variable is assigned the term with the corresponding index of the second, as follows.

Let the variables be $u = \{u_0, u_1, \ldots, u_N\}$ and $v = \{v_0, v_1, \ldots, v_N,\}$, the designation of the pair (u, v) establishes the multivalued functional relationship $u_0 \longrightarrow v_0, u_1 \longrightarrow v_1, u_2 \longrightarrow v_2, \ldots, u_N \longrightarrow v_N$. The relationship can also be one-valued. The ordered pair of the two variables can be symbolized as $(u, v) \longleftrightarrow v = f(u) \longleftrightarrow v_i = f(u_i)$. The letter f is a symbol or function letter and does not represent either of the two variables but the relationship that has been established between them. When the function letter is used, it is understood that it represents the pair of variables of the domain and the range.

For example, let $u = \{u_0, u_1, \ldots, u_N\}$ and $v = \{v_0, v_1, \ldots, v_N,\}$, where $u_n = n$ and $v_n = \sqrt{n}$. The function $v = f(u)$ is denoted, following the tradition $f(n) = \sqrt{n}$, although you can also write $v_n = \sqrt{n} = f(n) = f(u_n)$.

Otherwise, let $v = u^2$. The pair u, v is a function. The expression $v_n = f(u_n) = \left\{u_0^2, u_1^2, u_2^2, \ldots, u_N^2\right\}$ represents the terms of the corresponding sequences. The function $v = f(u)$ is given by $v = f(u) = u^2$. This can be written directly $f(u) = u^2$, although it lends itself to confusing the function f with the variable v.

When the variables used in the functional relationship are hyperreal, we have a hyperreal-valued function. Again, due to the Robinson transfer principle, there is a real-valued function equivalent to that, which is obtained by taking the standard part of the hyperreal function. Or, equivalently, taking the standard part of each of the elements of the variables involved.

3.2.4 Continuity of functions

The concept of continuity of the variable can be extended to the function, establishing the condition that one of the variables is independent. A function $v = f(u)$ is continuous when the variable v is continuous. In other words, f is continuous, if and only if dv is a variable that takes infinitesimal values.

Let $u = \{u_n\}$, $v = \{v_n\}$, for $0 \leq n \leq N$, where N is a fixed infinite hyperantor. The ordered pair (u, v) determines a continuous function if both variables are continuous. In other words, the function $v = f(u)$ is continuous if $du \approx 0 \to dv \approx 0$.

3.2.5 Integrals

Let the hyperreal finite interval $[-T, T]$, where the variable x is considered, which is the familiar sequence $x = \{ndx\}$ and $dx = T/N$, for $-N \leq n \leq N$, where N is hyperinteger infinity and dx is fixed infinitesimal. It is known that x is a continuous and independent variable on the interval $[-T, T]$.

Let $y = \{y_n\}$ be another variable, for $-N \leq n \leq N$. The product of the two variables is made, obtaining the new variable $ydx = \{y_n dx\}$. The quantity $y_n dx$ represents geometrically the infinitesimal area of the rectangle whose height is y_n and its base is the constant quantity dx, assuming that y_n is finite hyperreal. The integral of the variable ydx is a new variable, determined by the extremes of the interval $[-T, T]$, which can be written

$$\int_{-T}^{T} ydx = \sum_{-N}^{N} y_n dx. \tag{21}$$

The operation of adding hyperreal quantities in the indicated form is legitimate, although here the summation symbol expresses an infinite sum of hyperreal addends.

A special case occurs if $T \in \mathbb{R}$ is a fixed number, and the summation is a finite hyperreal. In this case, there is a standard part of the sum. If the calculation of the standard part does not depend on the choice of N, then it will be called definite integral given by

$$\int_{-T}^{T} ydx = st(\sum_{-N}^{N} y_n dx). \tag{22}$$

The previous situation does not change if the interval is not symmetric, but the arbitrary interval, $[a, b]$, where the extremes are real numbers, and the infinite hyperinteger N is chosen. In this case, the infinitesimal is $\alpha = (b - a)/N$ and the general term of the sequence is $x_n = a + n(b - a)/N$, for $0 \leq n \leq N$. Therefore, the independent variable is $x = \{x_n\} = \{n\alpha + a\}$ and its differential $dx = \{\alpha\}$.

Definition 8 *Let N be an infinite hyperinteger and the infinitesimal $\alpha = (b-a)/N$. Then, the integral defined on the interval $[a,b] \in \mathbb{R}$ is*

$$\int_a^b y dx = st(\sum_0^N y_n \alpha). \tag{23}$$

To guarantee that the sum in (23) is a finite hyperreal, the variable y must be finite and bounded, that is, $|y_n| \leq B$, for some finite B, since

$$\begin{aligned} |\int y dx| =& |\sum_0^N y_n dx| \leq \sum_0^N |y_n dx| \leq \sum_0^N |y_n| dx \leq \sum_0^N |B| dx \leq B dx \sum_0^N 1 \\ =& BN(T/N) = BT, \end{aligned}$$

then the sum is a finite hyperreal.

4 Converting the discrete sampling theorem to continuous sampling theorem

At this point, we are going to assume that variables and functions are hyperreal-valued. This means that both variables and functions can take finite, infinite and infinitesimal values. Among all possible combinations of T and N, let us first consider finite hyperreal T and infinite integer N. In this case, T/N is infinitesimal and determines the variable $t = \{ndt\}$, where $dt = T/N$ is a differential of t that is now an independent and continuous variable. There has been a conversion from the discrete domain $\{nT/N\}$ to the temporal continuous domain $\{ndt\}$, while f is the frequency, which is a discrete independent variable, because $df = 1/2T$ is not infinitesimal. However, the discrete domain at frequency $f = \{k/2T\}$ has infinite terms, since N is infinite.

We are going to analyze what happens with the exponential $e^{[n,k]}$ in the domains $\{ndt\}$, $\{k/2T\}$, for $-N \leq n, k < N$. It is clear that

$$e^{[j2\pi \frac{nk}{2N}]} = e^{[j2\pi (\frac{nT}{N})(\frac{k}{2T})]}. \tag{24}$$

Since $t = \{ndt\}$ and $dt = T/N$, we have

$$e^{[j2\pi \frac{nk}{2N}]} = e^{[j2\pi ndt(\frac{k}{2T})]} = e^{[j2\pi ndt f_k]}. \tag{25}$$

The transformation from the discrete complex exponential to the time continuous domain complex exponential has occurred. The complex exponential is still periodic over the temporal domain, but the periodicity in the frequency domain has been lost, because $N \leq k < N$ now runs through infinite frequency positions, since N is an infinite integer.

We now consider T and N, both infinite hyperreal, such that T/N remains infinitesimal. Not only is $dt = T/N$ infinitesimal, but $df = 1/2T$ is infinitesimal, therefore $f = \{kdf\}$ is a continuous variable.

In summary, we have obtained two domains $t = \{ndt\}$ and $f = \{kdf\}$ with continuous variables time and frequency. The complex exponential becomes

$$e^{[j2\pi \frac{nk}{2N}]} = e^{[j2\pi tkdf]} = e^{(j2\pi tf)}. \tag{26}$$

The transformation of the continuous complex exponential in time to the complex exponential of continuous domain in the frequency has occurred. The periodicity in the time and frequency domain has been lost, and we simply have the exponential complex $e^{(j2\pi tf)}$.

If we look at the real and imaginary parts of the exponential, the transformation of the discrete sinusoid appears. Specifically, we have shown that the following transformation occurs from the discrete to the continuous

$$\sin\left[2\pi\frac{nk}{2N}\right] \to \sin[2\pi ndtkdf] \to \sin(2\pi tf). \tag{27}$$

Let $x(t)$ be a function that satisfies the Shannon conditions, that is, defined on the real line and limited in band, whose bandwidth W is finite.

We can consider an interval of time $[-T, T]$ that covers the real line, where T is infinite. Let N be an infinite integer such that T/N is infinitesimal. From the point of view of infinitesimal calculus, the frequency occurs on the continuous domain $\{kdf\}$, where the infinitesimal df is the unit of frequency, with the corresponding continuous time domain ndt. Let us choose the hyperreal quantities so that $W = M/2T$, for infinite integer M of the same order of magnitude of T, so that the bandwidth W remains finite. With this bandwidth, we periodically extend the frequency spectrum P times, with infinite integer P, where $N = MP$.

While assuming both domains, time and frequency to be continuous, we will be careful to follow the transformation rule of the exponential complex already developed. The discrete version of the sampling theorem is

$$x\left[\frac{nT}{N}\right] = \frac{1}{2N}\sum_{q=-M}^{M-1} x\left[\frac{q}{2W}\right]\sum_{k=M}^{M} e^{j2\pi\frac{(n-qP)k}{2N}}. \tag{28}$$

By having assumed the domains $t = \{ndt\}$ and $f = \{kdf\}$, we have

$$x\left[ndt\right] = \frac{1}{P}\frac{1}{2M}\sum_{q=-M}^{M-1} x\left[\frac{q}{2W}\right]\sum_{k=M}^{M} e^{j2\pi\frac{(n-qP)k}{2N}}. \tag{29}$$

Now, since $(1/2M) = (1/2W)(1/2T)$ then

$$\frac{(n-qP)k}{2N} = \left(ndt - \frac{q}{2W}\right)kdf \tag{30}$$

then

$$x\left[ndt\right] = \frac{1}{2W}\sum_{q=-M}^{M-1} x\left[\frac{q}{2W}\right]\sum_{k=M}^{M} e^{j2\pi\left(ndt-\frac{q}{2W}\right)kdf}df. \tag{31}$$

Here $x\left[ndt\right]$ is a function of hyperreal value, as was initially assumed, because the variables (time and frequency) and the parameters that allow to define them, are hyperreal-valued. To obtain a real-valued function, such as those that represent signals in engineering, the standard part of the hyperreal-valued function must be taken.

$$\begin{aligned} x\left(t\right) =& st\left(x\left[ndt\right]\right) \\ =& st\left(\frac{1}{2W}\sum_{q=-M}^{M-1} x\left[\frac{q}{2W}\right]\sum_{k=M}^{M} e^{j2\pi\left(ndt-\frac{q}{2W}\right)kdf}df\right) \\ =& st\left(\frac{1}{2W}\right)\sum_{q=-\infty}^{\infty} st\left(x\left[\frac{q}{2W}\right]\right) st\left(\sum_{k=M}^{M} e^{j2\pi\left(ndt-\frac{q}{2W}\right)kdf}df\right), \end{aligned} \tag{32}$$

where $x(t)$ is a real-valued function, therefore time and frequency variables must be real-valued, as well as the parameters that define them. Then, $st\left(1/2W\right) = 1/2W$ and $st\left(x\left[q/2W\right]\right) = x\left[q/2W\right]$, that take real values for $W \in \mathbb{R}$.

The first sum on the right is a series of real numbers, while the standard part of last sum is precisely an integral defined on the interval $[-W, W] \in \mathbb{R}$ converted from the field $^*\mathbb{R}$ by application of the transfer principle. Therefore

$$x\left(t\right) = \frac{1}{2W}\sum_{q=-\infty}^{\infty} x\left[\frac{q}{2W}\right]\int_{-W}^{W} e^{j2\pi\left(t-\frac{q}{2W}\right)f}df. \tag{33}$$

The last integral has a known solution, so we will put the solution directly

$$x\left(t\right) = \sum_{q=-\infty}^{\infty} x\left[\frac{q}{2W}\right]\frac{\sin\pi\left(2Wt-q\right)}{\pi\left(2Wt-q\right)}. \tag{34}$$

This is the theorem of sampling in the continuous domain, which culminates our proof.

5 Conclusions

Discrete-time signal processing, also known as digital signal processing, emerged in the mid-20th century as a way to computationally simulate the continuous phenomena of nature. The mechanism for converting continuous to discrete phenomena was provided by Shannon with his sampling theorem. Since then, digital signal processing has continued as an independent discipline, with its own methods, without any link with the counterpart continues. Among these methods is the discrete sampling theorem, developed without any reference to Shannon's sampling theorem. This theorem has a discrete origin as a contribution to the improvement of other essentially discrete methods, such as the discrete Fourier transform. However, in engineering a wide variety of continuous phenomena are studied by discrete modeling. Therefore, from an educational point of view, there is a need to link the methods and findings of continuous and discrete processing. In this paper it was shown that when the theorems on sampling are placed on the basis of the NSA, they are equivalent. In practice, both formulations are two manifestations at different scales (finite and infinitesimal) of the same theorem. On a finite scale, the phenomena, their variables and functions are discrete in nature, and this is where the discrete sampling theorem is found. But on an infinitesimal scale, the phenomena, their variables and functions become continuous, and this is where Shannon's sampling theorem arises.

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