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LIE GROUP SYMMETRIES' COMPLETE CLASSIFICATION FOR A GENERALIZED CHAZY EQUATION AND ITS EQUIVALENCE GROUP

CLASIFICACIÓN COMPLETA DEL GRUPO DE SIMETRÍAS DE LIE PARA UNA ECUACIÓN DE CHAZY GENERALIZADA Y SU GRUPO DE EQUIVALENCIA

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Abstract

In this work, a complete classification of the Lie group symmetries for a generalization of Chazy equation was carried out and the equivalence group for the generalized Chazy equation is calculated and used to present the principal algebra of the equation.

Keywords: Lie symmetries; equivalence group; Lie symmetries classification; Chazy generalized equation .

Resumen

En este trabajo se obtiene una clasificación completa del grupo de simetrías de Lie para una generalización de la ecuación de Chazy, se calcula el grupo de equivalencia y se utiliza éste para presentar el álgebra principal de la ecuación.

Palabras clave: simetrías de Lie; grupo de equivalencia; clasificación de simetrías de Lie; ecuación generalizada de Chazy.

Mathematics Subject Classification: 76M60, 70G65, 34C14

1 Introduction

In [3], [8], [12], [9] it is introduced Prandtl's boundary layer equation for the stream function for an incompressible, steady two-dimensional flow with uniform or vanishing mainstream velocity as

$$\nu u_{yyy} = u_y u_{xy} - u_x u_{yy}, \tag{1}$$

where ν is a real number and represent the kinematic viscosity, the mentioned authors also introduce the following similarity transformation

$$u(x,y) = x^{1-\alpha} g(\omega), \quad \omega = \frac{y}{x^\alpha}\,, \tag{2}$$

where α is a real number. These authors affirm that using (2) in (1) the following non-linear third-order differential equation is obtained

$$\nu g_{\omega\omega\omega} + D g g_\omega + A(g_\omega)^2 = 0, \tag{3}$$

where $g(\omega)$ is a real function and at least $C^{(3)}$ and $D = 1-\alpha$ (2-dimensional form) or $D = 2-\alpha$ (radial form), with $A = 2-\alpha$ in both cases. In [9], the similarity transformation was made explicitly, that is, the author details all the calculations.

In [3], [12], a rough historical approach in different areas of mathematics and physics is presented for the equation (3).

In [27], Schlichting proposes a numerical solution for a free 2-dimensional jet in which $\alpha = 2/3$ and later an analytic solution was derived by Bickley [4]. In [28], Squire obtained the solution for the free radial jet for which $\alpha = 1$. In [13], Glauert presents solutions in parametric form when $\alpha = \frac{5}{4}$ (2-dimensional) and $\alpha = \frac{3}{4}$ (radial), and in [26], Riley presents a solution with $\alpha = 2$ (radial).

The case with $\alpha = -1$ (2-dimensional), $\alpha = -4$ (radial) and $\nu = 1$ has been the most studied equation using the theory of symmetries. In this case (3) becomes

$$y_{xxx} - 2yy_{xx} + 3y_x^2 = 0\,. \tag{4}$$

This equation, known as Chazy equation, was introduced by the same author in [10]. On the other hand, continuing with the use of Lie symmetries for the equation (4), in [11] Chazy established that the group of symmetries of (4) is a non-solvable group, and determined a reduction of this equation. In [12] Clarkson and Olver also established that the group of symmetries of (4) is a non-solvable algebra, and expressed the general solution of (4) as the ratio of two solutions of a hypergeometric equation. In [17], Ibragimov and Nucci, using the theory of Lie symmetries, were able to reduce (4) by the method of semi-canonical variables. In [1] the theory of non-local Lie symmetries was used to reduce (4), and later in [23] an improvement was presented. In [2], Arrigo calculated the group of symmetries and the invariant transformation of this group using canonical variables. Recently, in [20], [22], the authors characterized invariant solutions for the equation (4) and Kummer-Schwarz equation, from the generator operators of the optimal system, and using it, they presented new solutions.

In this line, given a differential equation, we can be considered by giving a generalization of it, defining coefficients from a differential equation in terms of parameters, obtained a family of EDS. Within this context, a problem of interest is to classify said family, into equivalence classes, according to the equivalence relation that defines each pair of differential equations of the family as equivalent if and only if, both differential equations have the same group of Lie symmetries. In this way, the family is divided into equivalence classes and is called a *complete classification of Lie symmetries group* (either the description of the classes or the description of the respective groups of symmetries), *this allows knowing all the differential equations of the family that have the same group of Lie symmetries for the initial equation*.

Besides, an equivalence transformation for a family the differential equations is a transformation of change of variables, which makes each member of the family correspond to a differential equation equivalent. The set of equivalence's transformations, endowed with the binary operation defined by the Lie bracket, is called *equivalence group of the differential equations family*. This type of study has been carried out in [23] for the Chazy equation, considering the coefficients as constant functions.

In this work, we extend the discussion started in [23], considering the following generalization of (4):

$$y_{xxx} - f(y)y_{xx} + 3y_x^2 = 0, \tag{5}$$

where f is an arbitrary function. Then, our objectives are: i) to provide a complete classification of Lie symmetries group for (5), ii) to determine the equivalence group of (5), and finally, iii) to illustrate the application of the equivalence group to obtain principal algebra.

2 Lie symmetries

In this section, we study the Lie point symmetries of (5). Following the classical Lie technique for calculating the symmetries of differential equations [5, 7, 18, 25], we carried out the complete group classification for (5). The corresponding result can be stated as follows:

Proposition 1 *The Lie point symmetry group of the generalized Chazy equation* (5) *with arbitrary f, is generated by*

$$X_1 = \partial_x. \ \ \textit{Principal algebra,} \tag{6}$$

other complementary cases are presented in the Table 1.

Proof. A general form of the one-parameter Lie group admitted by (5) is given by

$$x \to x + \epsilon\xi(x, y) + \cdots \quad \textit{and} \quad y \to y + \epsilon\eta(x, y) + \cdots ,$$

where ϵ is the group parameter. The vector field associated with the group of transformations shown above can be written as $\Gamma = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}$. Applying its third prolongation

$$\Gamma^{(3)} \quad = \quad \Gamma + \eta_{[x]}\frac{\partial}{\partial y_x} + \eta_{[xx]}\frac{\partial}{\partial y_{xx}} + \eta_{[xxx]}\frac{\partial}{\partial y_{xxx}},$$

Table 1: Infinitesimal generators of (5).

Case	f(y)	Condition	Infinitesimal generators of the group
$o)$	$f(y)$	$f_{yy}(y)\neq 0$	$X_1=\partial_x\ (Principal\ algebra).$
$i)$	$f(y)=yk_1+k_2$	$k_1,k_2\neq 0$	$X_1,\ \ X_2=-x\partial_x+\left(y+\frac{k_2}{k_1}\right)\partial_y.$
$ii)$	$f(y)=2y+k_2$	$k_2\neq 0$	$X_1,\ X_2=-x\partial_x+\left(y+\frac{k_2}{2}\right)\partial_y,$ $X_3=-\frac{x^2}{2}\partial_x+\left(yx+\frac{k_2}{2}+3\right)\partial_y.$
$iii)$	$f(y)=2y$	---	$X_1,\ X_2=-x\partial_x+y\partial_y,$ $X_3=-\frac{x^2}{2}\partial_x+(yx+3)\partial_y.$
$iv)$	$f(y)=k_2$	$k_2\neq 0$	$X_1,\ X_2=\partial y.$
$v)$	$f(y)=yk_1$	$k_1\neq 0,2$	$X_1,\ X_2=x\partial_x-y\partial_y.$

to (5), we must find the infinitesimals $\xi(x,y)$, $\eta(x,y)$ satisfying the symmetry condition

$$\eta(-f_y(y)y_{xx})+\eta_{[x]}(6y_x)+\eta_{[xx]}(-f(y))+\eta_{[xxx]}=0, \tag{7}$$

associated with (5). Here, $\eta_{[x]},\eta_{[xx]}$ and $\eta_{[xxx]}$ are the coefficients in $\Gamma^{(3)}$ given by

$$\begin{aligned}
\eta_{[x]} &= D_x[\eta]-(D_x[\xi])y_x=\eta_x+(\eta_y-\xi_x)y_x-\xi_y y_x^2,\\
\eta_{[xx]} &= D_x[\eta_{[x]}]-(D_x[\xi])y_{xx}\\
&= \eta_{xx}+(2\eta_{xy}-\xi_{xx})y_x+(\eta_{yy}-2\xi_{xy})y_x^2-\xi_{yy}y_x^3\\
&\quad +(\eta_y-2\xi_x)y_{xx}-3\xi_y y_x y_{xx}\\
\text{and } \eta_{[xxx]} &= D_x[\eta_{[xx]}]-(D_x[\xi])y_{xxx}\\
&= \eta_{xxx}+(3\eta_{xxy}-\xi_{xxx})y_x+3(\eta_{xyy}-\xi_{xxy})y_x^2\\
&\quad +\left(\eta_{yyy}-3\xi_{xyy}\right)y_x^3-\xi_{yyy}y_x^4+3\left(\eta_{xy}-\xi_{xx}\right)y_{xx}\\
&\quad +3\left(\eta_{yy}-3\xi_{xy}\right)y_x y_{xx}-6\xi_{yy}y_x^2 y_{xx}-3\xi_y y_{xx}^2\\
&\quad +\left(\eta_y-3\xi_x\right)y_{xxx}-4\xi_y y_x y_{xxx};
\end{aligned} \tag{8}$$

where D_x is the total derivative operator: $D_x = \partial_x + y_x\partial_y + y_{xx}\partial_{y_x} + y_{xxx}\partial_{y_{xx}} \cdots$. Replacing (8) into (7) we obtain:

$$\begin{aligned}
&\eta(-f_y(y)y_{xx}) + (6y_x)(\eta_x + (\eta_y - \xi_x)y_x - \xi_y y_x^2)\\
&-(f(y))(\eta_{xx} + (2\eta_{xy} - \xi_{xx})y_x + (\eta_{yy} - 2\xi_{xy})y_x^2 - \xi_{yy}y_x^3)\\
&-(f(y))((\eta_y - 2\xi_x)y_{xx} - 3\xi_y y_x y_{xx})\\
&+\eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y_x + 3(\eta_{xyy} - \xi_{xxy})y_x^2\\
&+(\eta_{yyy} - 3\xi_{xyy})\,y_x^3 - \xi_{yyy}y_x^4 + 3\,(\eta_{xy} - \xi_{xx})\,y_{xx}\\
&+3\,(\eta_{yy} - 3\xi_{xy})\,y_x y_{xx} - 6\xi_{yy}y_x^2 y_{xx} - 3\xi_y y_{xx}^2\\
&+(\eta_y - 3\xi_x)\,y_{xxx} - 4\xi_y y_x y_{xxx} = 0.
\end{aligned}$$

If we denote $f \cong f(y)$ and substitute $y_{xxx} = fy_{xx} - 3y_x^2$ in the last expression, then we can rearrange the expression with respect to $1, y_x, y_x^2, y_x^3, y_x^4, y_x y_{xx}$, $y_x^2 y_{xx}, y_{xx}$ and y_{xx}^2 for obtain

$$\begin{aligned}
&\eta_{xxx} - f\eta_x + (6\eta_x - f(2\eta_{xy} - \xi_{xx}) + 3\eta_{xxy} - \xi_{xxx})y_x\\
&(6\eta_y - 6\xi_x - f(\eta_{yy} - 2\xi_{xy}) + 3\eta_{xyy} - 3\xi_{xxy} - 3(\eta_y - 3\xi_x))y_x^2\\
&+(-6\xi_y + f\xi_{yy} + \eta_{yyy} - 3\xi_{xyy} + 12\xi_y)y_x^3 - \xi_{yyy}y_x^4\\
&+(3f\xi_y + 3\eta_{yy} - 9\xi_{xy} - 4f\xi_y)y_x y_{xx} - 6\xi_{yy}y_x^2 y_{xx}\\
&+(-f_y\eta - f(\eta_y - 2\xi_x) + 3\eta_{xy} - 3\xi_{xx} + f(\eta_y - 3\xi_x))y_{xx}\\
&-3\xi_y y_{xx}^2 = 0.
\end{aligned}$$

As we know, the variables $1, y_x, y_x^2, y_x^3, y_x^4, y_x y_{xx}, y_x^2 y_{xx}, y_{xx}$ and y_{xx}^2, are linearly independent, thus the previous expression we obtain the determining equations

$$\begin{aligned}
\xi_y = \eta_{yy} &= 0, && \text{(9a)}\\
\eta_y + \xi_x &= 0, && \text{(9b)}\\
-f\eta_{xx} + \eta_{xxx} &= 0, && \text{(9c)}\\
f(-2\eta_{xy} + \xi_{xx}) + 6\eta_x + 3\eta_{xxy} - \xi_{xxx} &= 0, && \text{(9d)}\\
-f_y\eta - f\xi_x + 3\eta_{xy} - 3\xi_{xx} &= 0. && \text{(9e)}
\end{aligned}$$

Solving in (9a), we have $\xi = c_1(x)$ and $\eta = yc_2(x) + c_3(x)$, with c_1, c_2, c_3 as arbitrary functions. From the expressions for ξ and η into (9b), (9c), (9d) and (9e), we obtain respectively:

$$\begin{aligned}
c_2(x) + c_1'(x) &= 0, && \text{(10a)}\\
-(yc_2''(x) + c_3''(x))f + yc_2'''(x) + c_3'''(x) &= 0, && \text{(10b)}\\
[-2c_2'(x) + c_1''(x)]f + 6[yc_2'(x) + c_3'(x)] + 3c_2''(x) - c_1'''(x) &= 0, && \text{(10c)}\\
-[yc_2(x) + c_3(x)]f_y - c_1'(x)f + 3c_2'(x) - 3c_1''(x) &= 0. && \text{(10d)}
\end{aligned}$$

Using (10a) and differentiating (10c) with respect to y we obtain $-3c_2'(x)f_y + 6c_2'(x) = 0$, whose derivative with respect to y yields

$$-3c_2'(x)f_{yy} = 0. \tag{11}$$

From (11), we consider two cases: $f_{yy} \neq 0$ or $f_{yy} = 0$.

First, we consider the case $f_{yy} \neq 0$. According to (11), we can assert that $-3c_2'(x) = 0$, hence $c_2(x) = k_1$, with k_1 an arbitrary constant. Then from (10a), we obtain:

$$c_2(x) = k_1, \qquad c_1(x) = -xk_1 + k_2, \tag{12}$$

where k_2 is an arbitrary constant. By replacing (12) into (10c), we get $c_3'(x) = 0$, so that $c_3(x) = k_3$. From the above and (12) into (10d), it follows that $-(yk_1 + k_3)f_y + f(k_1) = 0$. By differentiating in the previous expression with respect to y, we have $(yk_1 + k_3)f_{yy} = 0$. But we know that $f_{yy} \neq 0$, therefore $k_1 = 0$ and $k_3 = 0$. Consecuently, from $c_3(x) = k_3$ and (12), we obtain the infinitesimal generators of the group, $\xi = c_1(x) = k_2$ and $\eta = k_1 y + k_3 = 0$, which is equivalent to $X_1 = \partial_x$ with $f_{yy} \neq 0$.

We consider now the case $f_{yy} = 0$, and proceed to use (11) to establish (6). First, from $f_{yy} = 0$ we have

$$f(y) = yk_1 + k_2, \tag{13}$$

with k_1, k_2 as arbitrary constants. From (13), we see that there are two cases to be considered regarding $k_1 \neq 0$ or $k_1 = 0$.

1. Case $k_1 \neq 0$. For (13) consider two cases: $k_2 \neq 0$ or $k_2 = 0$.

   (a) Case $k_1 \neq 0$ and $k_2 \neq 0$. From (13) and (10b) we get $-(yk_1 + k_2)(yc_2''(x) + c_3''(x)) + yc_2'''(x) + c_3'''(x) = 0$. By differentiating the last expression with respect to y, we obtain

   $$-k_2c_2''(x) - 2k_1yc_2''(x) - k_1c_3''(x) + c_2'''(x) = 0. \tag{14}$$

   Now, by differentiating (14) with respect to y, we find $-2k_1c_2''(x) = 0$, that implies $c_2'(x) = k_3$, and thus, $c_2(x) = xk_3 + k_4$, where k_3, k_4 are arbitrary constants. From (10a) we have

   $$c_1(x) = -\frac{x^2}{2}k_3 - xk_4 + k_5. \tag{15}$$

By substitution of $c_2(x) = xk_3 + k_4$ into (14), we get $c_3''(x) = 0$, so

$$c_3(x) = xk_6 + k_7. \tag{16}$$

From (13), $c_2(x) = xk_3 + k_4$, (15) and (16) into (10c), we obtain

$$(yk_1 + k_2)(-3k_3) + 6(yk_3 + k_6) = 0, \tag{17}$$

and by differentiating this expression with respect to y, we find

$$k_3(-k_1 + 2) = 0. \tag{18}$$

In last equation we see that there are two cases to be considered: $k_3 = 0$ or $k_3 \neq 0$.

i. Case $k_1 \neq 0$, $k_2 \neq 0$, and $k_3 = 0$. From (17) we have $k_6 = 0$. From $c_2(x) = k_4$, (15) and (16), we have:

$$c_1(x) = -xk_4 + k_5, \quad c_2(x) = k_4, \quad c_3(x) = k_7. \tag{19}$$

From (19) and (13) into (10d), we find $-k_1k_7 + k_2k_4 = 0$, implies $k_7 = \frac{k_2k_4}{k_1}$. Then, we can rewrite (19) as :

$$c_1(x) = -xk_4 + k_5, \quad c_2(x) = k_4, \quad c_3(x) = \frac{k_2k_4}{k_1}.$$

Summarizing, the infinitesimal generators of the group are: $\eta = yk_4 + \frac{k_2k_4}{k_1}$, $\xi = -xk_4 + k_5$, which is equivalent to:

$$X_1 = \partial_x \quad \text{and} \quad X_2 = -x\partial_x + \left(y + \frac{k_2}{k_1}\right)\partial_y.$$

ii. Case $k_1 \neq 0$, $k_2 \neq 0$ and $k_3 \neq 0$. From (18), we have $k_1 = 2$. From (17), $(2y + k_2)(-3k_3) + 6(yk_3 + k_6) = 0$, which implies $k_6 = \frac{k_3k_2}{2}$. From the above, $c_2(x) = xk_3 + k_4$, (15) and (16), we obtain

$$\begin{aligned} c_1(x) &= -\frac{x^2}{2}k_3 - xk_4 + k_5, \qquad c_2(x) = xk_3 + k_4,\\ c_3(x) &= x\frac{k_3k_2}{2} + k_7. \end{aligned} \tag{20}$$

Inserting (13) and (20) into (10d), we find $-2[y(xk_3 + k_4) + (x\frac{k_3k_2}{2} + k_7)] - (2y + k_2)[-(xk_3 + k_4)] + 6k_3 = 0$, hence

$k_7 = \frac{k_2 k_4}{2} + 3k_3$.
From this into (20) it follows that $c_1(x) = -\frac{x^2}{2}k_3 - xk_4 + k_5$, $c_2(x) = xk_3 + k_4, c_3(x) = x\frac{k_3 k_2}{2} + \frac{k_2 k_4}{2} + 3k_3$.

Summarizing, the infinitesimal generators of the group are:

$$\eta = yxk_3 + yk_4 + x\frac{k_3 k_2}{2} + \frac{k_2 k_4}{2} + 3k_3,$$
$$\xi = -\frac{x^2}{2}k_3 - xk_4 + k_5,$$

which is equivalent to $X_1 = \partial_x$, $X_2 = -x\partial_x + (y + \frac{k_2}{2})\partial_y$, and

$$X_3 = -\frac{x^2}{2}\partial_x + (yx + \frac{k_2}{2} + 3)\partial_y.$$

(b) Case $k_1 \neq 0$ and $k_2 = 0$. Since (13), we have $f(y) = yk_1$, then using (10b), we have $-y^2(k_1 c_2''(x)) - y(k_1 c_3''(x)) - c_2'''(x)) + c_3'''(x) = 0$. By differentiating this with respect to y, it follows

$$-2yk_1(c_2''(x)) - k_1 c_3''(x) + c_2'''(x) = 0. \tag{21}$$

From this, we conclude that $c_2''(x) = 0$, thus $c_2(x) = xk_3 + k_4$, with k_3, k_4 as arbitrary constants. When replacing $c_2(x) = xk_3 + k_4$ into (21), we obtain $c_3''(x) = 0$, hence $c_3(x) = xk_6 + k_7$, with k_6, k_7 as arbitrary constants. From $c_2(x) = xk_3 + k_4$ into (10a), we find $c_1'(x) = -xk_3 - k_4$, hence $c_1(x) = -\frac{k_3}{2}x^2 - xk_4 + k_5$. Now, from $f(y) = yk_1$, $c_2(x) = xk_3 + k_4$, $c_3(x) = xk_6 + k_7$ and $c_1(x) = -\frac{k_3}{2}x^2 - xk_4 + k_5$ into (10c), we obtain $yk_3(-k_1 + 2) + 2k_6 = 0$, which implies that

$$k_6 = 0 \qquad \text{and} \qquad k_3(-k_1 + 2) = 0. \tag{22}$$

Then, from (22) consider two cases: $k_3 = 0$ or $k_3 \neq 0$.

i. Case $k_1 \neq 0$, $k_2 = 0$ and $k_3 = 0$. From $c_2(x) = xk_3 + k_4$, $c_3(x) = xk_6 + k_7$ and $c_1(x) = -\frac{k_3}{2}x^2 - xk_4 + k_5$, we have

$$c_1(x) = -xk_4 + k_5, \quad c_2(x) = k_4 \text{ and } c_3(x) = k_7.$$

From the last expressions into (10d), we obtain $k_7 = 0$.
Then, we get: $c_1(x) = -xk_4 + k_5, c_2(x) = k_4$ and $c_3(x) = 0$.
Summarizing, the infinitesimal generators of the group are:
$\eta = yk_4$ and $\xi = -xk_4 + k_5$, which is equivalent to:

$$X_1 = \partial_x \text{ and } X_2 = x\partial_x - y\partial_y.$$

ii. Case $k_1 \neq 0$, $k_2 = 0$ and $k_3 \neq 0$. From (22), we have $k_1 = 2$. Then, from $c_2(x) = xk_3 + k_4$, $c_3(x) = xk_6 + k_7$, and $c_1(x) = -\frac{k_3}{2}x^2 - xk_4 + k_5$, we find: $c_1(x) = -\frac{k_3}{2}x^2 - xk_4 + k_5$, $c_2(x) = xk_3 + k_4$ and $c_3(x) = k_7$.
Summarizing, the infinitesimal generators of the group are: $\eta = yxk_3 + yk_4 + k_7,\ \xi = -\frac{k_3}{2}x^2 - xk_4 + k_5$, with $k_7 = 3k_3$, which is equivalent to:

$$X_1 = \partial_x, \quad X_2 = -x\partial_x + y\partial_y, \quad X_3 = -\frac{x^2}{2}\partial_x + (yx+3)\partial_y \,.$$

2. From (13), suppose that $k_1 = 0$. Then, we have $f(y) = k_2$ and consequently, from (10b) we obtain:

$$y[-k_2 c_2''(x) + c_2'''(x)] - k_2 c_3''(x) + c_3'''(x) = 0.$$

Using (10a) in the expression above we can conclude:

$$c_1(x) = \frac{-a_1}{k_2^3}e^{xk_2} - \frac{a_3}{2}x^2 - xa_2 + a_4, \;\; c_2(x) = \frac{a_1}{k_2^2}e^{xk_2} + xa_3 + a_2,$$

$$c_3(x) = \frac{a_5}{k_2^2}e^{xk_2} + xa_6 + a_7, \tag{23}$$

with $a_1, \cdots, a_7$ arbitrary constants. Using (23) into (10c) we get:

$$e^{xk_2}\left[y\left(\frac{6a_1}{k_2}\right) + a_1 + \frac{6a_5}{k_2}\right] + 6a_6 - 3a_3k_2 + y(6a_3) = 0,$$

which implies that $a_1 = a_3 = a_5 = a_6 = 0$. From (23), $c_1(x) = -xa_2 + a_4, c_2(x) = a_2$ and $c_3(x) = a_7$. From the last expressions into (10d), we have $k_2a_2 = 0$, hence $a_2 = 0$. Then, $c_1(x) = a_4$, $c_2(x) = 0$, and $c_3(x) = a_7$. Thus, the infinitesimal generators of the group are $\eta = a_7$ and $\xi = a_4$, which is equivalent to

$$X_1 = \partial_x \text{ and } X_2 = \partial_y.$$

■

Remark 1 *In [10], the generators of the symmetry group of Equation (4) are presented without showing details for the calculations, these details are presented in [20], where the optimal algebra and new invariant solutions are presented too. Now, the previous results for the generators of the Lie symmetries group coincide with the part (iii) of Proposition 1, after manipulating some constants, which are:* $X_1 = \partial_x, X_2 = x\partial_x - y\partial_y$ *and* $X_3 = x^2\partial_x - (2yx+6)\partial_y$.

3 The equivalence group

An equivalence transformation for (5) is a change of variables $(x, y) \to (\tilde{x}, \tilde{y})$, where the structure of (5) is conserved. The infinitesimal generators of the continuous group of equivalence transformations have the form

$$\Gamma \;=\; \xi(x,y)\partial_x + \eta(x,y)\partial_y + \mu(x,y,f)\partial_f. \tag{24}$$

According to [25, 15], the operator (24) generates the continuous equivalence group if it is admitted by the extended system:

$$\begin{cases} y_{xxx} - f y_{xx} + 3y_x^2 = 0, \\ f_x = 0. \end{cases} \tag{25}$$

The infinitesimal invariance test for the extended system (25) requires the following prolongation of the operator (24),

$$\begin{aligned} \Gamma^{(3)} \;=\;& \xi(x,y)\partial_x + \eta(x,y)\partial_y + \mu(x,y,f)\partial_f \\ & + \eta_{[x]}\partial_{y_x} + \eta_{[xx]}\partial_{y_{xx}} + \eta_{[xxx]}\partial_{y_{xxx}} + \omega\partial_{f_x}, \end{aligned}$$

where $\eta_{[x]}, \eta_{[xx]}$ and $\eta_{[xxx]}$ are given in (8), $\omega = \widetilde{D}_x[\mu] - f_x\widetilde{D}_x[\xi] - f_y\widetilde{D}_x[\eta]$ and $\widetilde{D}_x = \frac{\partial}{\partial x} + f_x\frac{\partial}{\partial f} + f_{xx}\frac{\partial}{\partial f_x}$. Applying $\Gamma^{(3)}$ to the extended system (25), we obtain that the invariance condition:

$$\begin{cases} \mu(-y_{xx}) + \eta_{[x]}(6y_x) + \eta_{[xx]}(-f) + \eta_{[xxx]} = 0, \\ \omega = 0. \end{cases} \tag{26}$$

Substituting (25) in (26), we have $\omega = \mu_x - f_y(\eta_x) = 0$. Now, as the constant 1 and f_y are algebraically independent, then first equation above implies $\mu_x = 0$ and $\eta_x = 0$. Thus $\mu = \mu(y, f)$, $\eta = \eta(y)$, and $\xi = \xi(x, y)$. With these in hand, we substitute (8) into (26) and get

$$\begin{aligned} & \mu(-y_{xx}) + (6y_x)((\eta_y - \xi_x)y_x - \xi_y y_x^2) \\ -\;& (f)((-\xi_{xx})y_x + (\eta_{yy} - 2\xi_{xy})y_x^2 - \xi_{yy}y_x^3 + (\eta_y - 2\xi_x)y_{xx} - 3\xi_y y_x y_{xx}) \\ -\;& (\xi_{xxx})y_x + 3(-\xi_{xxy})y_x^2 + (\eta_{yyy} - 3\xi_{xyy})\, y_x^3 - \xi_{yyy}y_x^4 + 3\,(-\xi_{xx})\, y_{xx} \\ & + 3\,(\eta_{yy} - 3\xi_{xy})\, y_x y_{xx} - 6\xi_{yy}y_x^2 y_{xx} - 3\xi_y y_{xx}^2 \\ & + (\eta_y - 3\xi_x)\, y_{xxx} - 4\xi_y y_x y_{xxx} = 0. \end{aligned}$$

now, we replace $y_{xxx} = y_{xx}f - 3y_x^2$ in the last expression.

As know, the variables $y_x, y_x^2, y_x^3, y_x^4, y_x y_{xx}, y_x^2 y_{xx}, y_{xx}, y_{xx}^2$ are linearly independent; hence we obtain the following system of determining equations

$$\xi_y = \eta_{yy} = \mu_x = \eta_x = 0, \tag{27a}$$

$$\eta_y + \xi_x = 0, \tag{27b}$$

$$f\xi_{xx} - \xi_{xxx} = 0, \tag{27c}$$

$$\mu + f\xi_x + 3\xi_{xx} = 0. \tag{27d}$$

From (27a), we get

$$\xi = c_1(x), \qquad \eta = yc_2 + c_3 \qquad \text{and} \qquad \mu = \mu(y, f), \tag{28}$$

where c_1 is arbitrary function and c_2, c_3 are arbitrary constants. From (28) into (27b) it follows $c_2 + c_1'(x) = 0$, which implies $c_1(x) = -c_2 x + c_4$. From the last expresion in (27d) we get $\mu = fc_2$, then the infinitesimal generators of the equivalence group of the (5) are $\eta = yc_2 + c_3$, $\xi = -c_2 x + c_4$, and $\mu = fc_2$ which is equivalent to $X_1 = -x\partial_x + y\partial_y + f\partial_f$, $X_2 = \partial_x$ and $X_3 = \partial_y$. We summarize the above considerations as follows:

Proposition 2 *The Lie algebra for the continuous equivalence group of Equation* (5)*, is generated by the following vector fields:*

$$X_1 = -x\partial_x + y\partial_y + f\partial_f, \qquad X_2 = \partial_x, \qquad X_3 = \partial_y. \tag{29}$$

4 Principal algebra using the equivalence group

In this section, we show the consistency of the previous results by using the equivalence group to calculate the principal algebra, which coincides with that obtained in the Proposition 1.

The most general symmetries algebra of Equation (5), for an arbitrary function f is called the Principal Lie Algebra $L_{\mathfrak{p}}$ [15, 19]. This algebra can be obtained from equivalence algebra $L_{\mathfrak{e}}$ (29). Let us denote by $\mathbf{x} = (x)$ and $\mathbf{y} = (y)$ the independent and dependent variables, respectively, and by $\mathbf{f} \equiv \{f\}$ the arbitrary elements in the system (5). Let the projection $X = pr_{(\mathbf{x},\mathbf{y})}(Y)$ of the (29) to the space $(\mathbf{x}, \mathbf{y})$ of the independent and dependent variables

$$X = pr_{(\mathbf{x},\mathbf{y})}(Y) = \xi\partial x + \eta\partial y. \tag{30}$$

Also, consider the projection $Z = pr_{(\mathbf{y},\mathbf{f})}(Y)$ to the space $(\mathbf{y}, \mathbf{f})$ involved in the arbitrary elements

$$Z = pr_{(\mathbf{y},\mathbf{f})}(Y) = \eta\partial y + \mu\partial f. \tag{31}$$

From the above, it is clear that operators X and Z are well defined, i.e. their coordinates involve only the respective variables $(\mathbf{x},\mathbf{y})$ and $(\mathbf{y},\mathbf{f})$. For our specific case, the Proposition on projections was formulated in [16, 19] as follows:

Proposition 3 *An operator X belongs to the principal Lie algebra $L_{\mathfrak{p}}$ for (5) if and only if $X = pr_{(\mathbf{x},\mathbf{y})}(Y)$, where Y is an equivalence generator such that $Z = pr_{(\mathbf{y},\mathbf{f})}(Y) \equiv 0$.*

Now, the idea is to apply Proposition 3 to find the principal Lie algebra $L_{\mathfrak{p}}$ of (5). Let‘s consider the equivalence generator $Y = k_1X_1 + k_2X_2 + k_3X_3$, which is a linear combination of the operators in (29). The above is equivalent to

$$Y = k_1[-x\partial x + y\partial y + f\partial f] + k_2[\partial_x] + k_3[\partial_y], \tag{32}$$

with projections (30) and (31) as

$$X = [-xc_2 + c_4]\partial x + [yc_2 + c_3]\partial y,$$

$$Z = [yc_2 + c_3]\partial y + [fc_2]\partial f.$$

Now, we know that $Z \equiv 0$ if and only if $c_2 = c_3 = 0$, thus applying Proposition 3, we can conclude that $Y = k_2X_2$. So, the principal Lie algebra $L_{\mathfrak{p}}$ of (5) is spanned by $\{X_2\}$, where X_2 is the presented in the Proposition 2 which coincides with (6), from Proposition 1.

Remark 2 *Note that the Proposition 3 can be proved with the following considerations: We recall that the principal Lie algebra consists of all the operators $\Gamma = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ admitted by equation (5) for any $f(y)$. Therefore the principal Lie algebra is the subalgebra of the equivalence algebra, such that any operator Y (in our case these operator are X_1, X_2, X_3) of this subalgebra leaves invariant equations $f = f(y)$. It follows that f and y are invariant with respect to Y. It means that $\eta = 0$ and $\mu = 0$, or $Pr(y,f)(Y) = 0$.*

5 Conclusions

In this work, the complete classification of the symmetry group of the generalized Chazy equation (5) was carried out (Proposition 1). According to Table 1, the family of differential equations defined by (5), where $f(y)$ is a parameter, the only equation that has the same group of symmetries as the Chazy equation (4), is itself.

Furthermore, in the item $ii)$ of the table 1 it can be noted that there is a more general Chazy equation that has the same group symmetries as the equation (4). In Proposition 1, the principal algebra is calculated and it coincides with that obtained by applying Proposition 3 to the equivalence group.

Ideas to develop in future works could be calculating the conservation laws for (4), using the symmetries presented in Proposition 1, also it is possible to investigate the contact symmetries as was presented in [24, 14, 6] and try to obtain for the symmetries associated with the equation (4), the classification of its Lie algebra as was presented in [21].

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Data availability

The data used to support the findings of this study are included within the article.

Conflicts of interest

The authors declare that they have no conflicts of interest.

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